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The need for high order numerical schemes to model dispersive high frequency acoustic waves in water-filled pipes

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ABSTRACT
Probing fluid pipelines with high frequency transient waves permits high resolution defect detection. For probing frequency above the first cut-off frequency, radial and/or azimuthal modes are excited and plane wave theory does not apply. A two dimensional explicit scheme based on the finite-volumes method uses an approximate Riemann solver to evaluate hyperbolic terms. To minimize numerical dissipation and reduce computations, higher order schemes using weighted essential non-oscillatory cell-reconstruction and the Runge–Kutta method for time evolution are developed and tested. The parabolic (viscous) part is spatially discretized by second-order finite differences and operator splitting (fourth-order Runge–Kutta) for simultaneous evaluation of inviscid and viscous parts. The characteristic boundary condition for dispersive high frequency waves is also studied. Radial wave propagation from rapid valve closure is discussed and results show the propagation of radial waves if closure time induced frequency is higher than cut-off frequencies.

Keywords: CFD; finite volume method; high frequency waves; high order scheme; pipe flow; transient flows

1 Introduction

In the last two decades, active research has increased regarding use of rapid fluid transients (acoustic waves in fluid) to detect defects in pipe systems (e.g. Duan, Lee, Ghidaoui, & Tung, 2011; Lee, Duan, Tuck, & Ghidaoui, 2015; Louati & Ghidaoui, 2015; Meniconi et al., 2013). These techniques are generally called transient-based defect detection methods (TBD-DMs). There is consensus that the accuracy of this technique increases as the probing wave frequency increases. However, from general acoustic wave theory, as the frequency of these waves increases beyond the cut-off frequencies of radial and/or azimuthal modes, acoustic waves in pipes become dispersive. In other words, the plane wave assumption that pressure is invariant across the pipe cross section becomes invalid. Therefore, classical 1D and quasi-2D water-hammer models cease to be valid.

In transient field, the most well-known and referenced work (radial variation of pressure is included) is by Mitra and Rouleau (1985), in which they employ an implicit numerical scheme based on matrix factorization. Specifically, Mitra and Rouleau (1985) use a finite difference method (FDM) for space discretization and a three backward point approach for time marching. In addition, a predictor-corrector technique sweeps along the axial and radial direction simultaneously. Mitra and Rouleau (1985) report that radial waves are only important near devices (e.g. valves) and singularities. Although they test sudden valve closure, their wave front shows excessive smearing from numerical dissipation. In addition, their scheme is time consuming (about 2 h for 1000 total mesh size on a single core) and test cases are limited to very short pipe lengths (a multiple of a few pipe diameters in length). Excessive numerical dissipation and computational time make this scheme unsuitable for transient high frequency waves (HFW).

Recent work by Wang et al. (2016) and Martins, Soares, Ramos, and Covas (2016) use implicit schemes in ANSYS which are also very dissipative. In fact, radial waves are not
observed in Wang et al. (2016) and Martins et al. (2016). Wang et al. (2016) mentioned that the advantage of using 2D models is only for the 2D visualization of cavity formation. It is not surprising that Wang et al. (2016) claim that the scheme is highly dissipative and cannot catch the true experimental magnitude. Martins et al. (2016) had to choose an optimal mesh size because the required fine mesh is very time consuming. Their results show no indication of radial waves although a sudden instantaneous valve closure is used. This clearly demonstrates that the scheme is highly dissipative to catch radial (high frequency) waves. This paper seeks to develop a scheme that is accurate, efficient and preserves the flow physics of HFW.

1.1 Properties of numerical schemes

Finite difference methods (FDM) (such as Lax–Wendroff schemes) should be avoided due to their high dissipative and dispersive behaviour (Hirsch, 1988b). Indeed, as explained above, the FDM scheme of Mitra and Rouleau (1985) is found to be highly dissipative and inefficient. Although advanced FDMs are developed (e.g. Bogey & Bailly, 2004; Lele, 1992; Tam & Webb, 1993) where numerical dispersion and dissipation are minimized, they are not appropriate for modelling the physical dispersion of HFW. In addition, FDMs are non-conservative, which make them non-suitable for transient flow modelling. Finite element schemes (Hirsch, 1988a) can efficiently handle complex boundary conditions, but they suffer from non-conservation issues and could induce numerical dispersion errors (Girault & Raviart, 2012). Recently, lattice Boltzmann schemes have become popular in the fluid dynamics community because of their simplicity and suitability for parallel computation. However, lattice Boltzmann schemes have low order accuracy (second order) and their application to HFW in pipes has proved to be inefficient (Louati, Tekitek, & Ghidaoui, 2017) thus far.

Finite volume (FV) schemes are conservative, highly suited for wave problems, and their order of accuracy can be easily increased. FV schemes can be broadly classified into Godunov-type schemes (GTS) (Toro, 2009) and gas kinetic schemes (GKS) (Chen & Doolen, 1998; Xu, 2014). The GTS use the Riemann solution (RS) to evaluate the numerical fluxes at the cell boundaries. While GKS use the Boltzmann equation to evaluate fluxes at finite volume interfaces. Between GTS and GKS by Li, Li, and Xu (2011) shows that the main advantages of GKS over GTS are (i) the ability of GKS to evaluate inviscid and viscous parts together in a single flux formulation without operator splitting and (ii) the order of accuracy of GKS is easier to increase using high order Taylor expansion (Luo, Xuan, & Xu, 2013; Xu, 2014). However, GKS are computationally about three times slower than GTS when the approximate Riemann solver is used (Li et al., 2011). In addition, it is noted that the advantages of GKS are beneficial only when solving nonlinear systems and shock-capturing problems (i.e. large Mach number problems). This is not the case for water-hammer problems because they are only weakly nonlinear (very low Mach number) and only involve jump discontinuities. Therefore, a GTS scheme based on approximate Riemann solution is developed and tested in this work.

The paper is organized as follows. Governing equations are given in Section 2. The details of the numerical schemes are provided in Section 3. In Section 4, different properties (stability, accuracy, dissipation and dispersion) of the scheme are studied. Section 5 studies the case of two-dimensional water hammer flow due to sudden valve closure and discusses the behaviour of radial waves. Conclusions are drawn in Section 6. Most of the mathematical details of the scheme are relegated to the Appendix, which includes the Riemann solution, the Hancock-approach, and the weighted essential non-oscillatory (WENO) reconstruction approach and the implementation of characteristic boundary conditions for modelling non-reflective boundaries.

2 Governing equations

Let x and r denote axial and radial coordinates; t is time; \( V_x \) and \( V_r \) denote macroscopic velocity components in x- and r-directions (Fig. 1), respectively; and \( \rho \) and \( P \) denote the fluid density and pressure, respectively. The 2D axisymmetric Navier–Stokes equations for a compressible and isothermal Newtonian fluid in cylindrical coordinate are (Ghidaoui, 2004):

\[
\frac{\partial U}{\partial t} + \frac{\partial F}{\partial r} + \frac{\partial G}{\partial x} = S
\]  

where:

\[
U = \begin{pmatrix} \rho \\ \rho V_r \\ \rho V_x \\ \rho V_s^2 + P \end{pmatrix} ; \quad F = \begin{pmatrix} \rho V_r \\ \rho V_r V_x \\ \rho V_r V_s^2 + P \end{pmatrix} ; \quad G = \begin{pmatrix} \rho V_x \\ \rho V_r V_x \\ \rho V_s^2 + P \end{pmatrix}
\]

\[
S = \begin{pmatrix} \frac{\rho V_r}{r} \\ \frac{\partial \tau_{rr}}{\partial r} + \frac{\partial \tau_{rx}}{\partial x} + \frac{1}{r} \left( \tau_{rr} - \tau_{xx} - \rho V_r^2 \right) \\ \frac{\partial \tau_{rx}}{\partial r} + \frac{\partial \tau_{xx}}{\partial x} + \frac{1}{r} \left( \tau_{xx} - \rho V_r V_x \right) \\ \frac{\partial \tau_{xs}}{\partial r} + \frac{\partial \tau_{sx}}{\partial x} + \frac{1}{r} \left( \tau_{xs} - \rho V_r V_s \right) \end{pmatrix}
\]  

Figure 1 Coordinate system in the pipe.
In Eq. (3), \( \tau_{rr}, \tau_{\theta\theta}, \tau_{\phi}\phi, \) and \( \tau_{r\theta} \) denote the components of the stress tensor given by:

\[
\begin{align*}
\tau_{rr} &= (2\mu + \lambda) \frac{\partial V_r}{\partial r} + \lambda \left( \frac{V_r}{r} + \frac{\partial V_r}{\partial r} \right) ; \\
\tau_{\theta\theta} &= (2\mu + \lambda) \frac{\partial V_\theta}{\partial \theta} + \lambda \left( \frac{V_\theta}{r} + \frac{\partial V_\theta}{\partial r} \right) ; \\
\tau_{\phi\phi} &= (2\mu + \lambda) \frac{\partial V_\phi}{\partial \phi} + \lambda \left( \frac{V_\phi}{r} + \frac{\partial V_\phi}{\partial r} \right) ; \\
\tau_{r\theta} &= \mu \left( \frac{\partial V_r}{\partial \theta} + \frac{\partial V_\theta}{\partial r} \right) .
\end{align*}
\]

Here \( \lambda = -2/3\mu \) and \( \mu \) denotes the dynamic viscosity of the fluid. At each line in Eq. (3), the last terms occur due to the cylindrical coordinate formulation.

Pressure is related to density using the following isentropic and isothermal state equation for an infinitely rigid conduit (Chaudhry, 2014; Ghidaoui, 2004; Wylie, Streeter, & Suo, 1993):

\[
\frac{\partial P}{\partial \rho} = a^2 = \frac{K}{\rho}
\]

where \( K \) is the bulk modulus of water and \( a \) is the wave speed (assumed to be constant). If the fluid is water, and the pipe is perfectly rigid, the wave speed \( a \) is about 1440 m s\(^{-1}\). Equation (5) can also be applied to slightly deformable pipes, where an effective bulk modulus of elasticity, which depends on pipe thickness, diameter, and Young’s modulus of elasticity, is used. For simplicity of calculation in this paper, \( a \) is taken to be 1000 m s\(^{-1}\).

### 3 Numerical scheme

In this section, a two-dimensional (2D) FV numerical scheme based on the approximate Riemann solver (Toro, 2009) is developed for water-hammer flow application, and its robustness and accuracy are tested for modelling high frequency (dispersive) waves in water-filled pipe flow. Second, third and fifth orders of accuracy are considered. The second-order accuracy in space and time is achieved using the monotone upstream-centred scheme (Hirsch, 1988b), and the classical third-order Runge–Kutta (RK) method is used for time marching (Hirsch, 1988b).

The developed numerical scheme solves the two-dimensional axisymmetric Navier–Stokes equations (Eqs (1)–(4)) for unsteady and compressible flow in water-filled pipe (Fig. 1). The physical domain is discretized into a numerical domain (Fig. 2) containing \( N_r \) and \( N_\theta \) \((N_r \times N_\theta)\) finite volumes along the radial and axial directions, respectively. As the pipe system is assumed to be axisymmetric, only the half-domain of the pipe need be considered \((0 \leq r \leq R)\) where \( R = D/2 \) is the pipe radius and \( D \) is the pipe diameter. Since the numerical scheme used is based on FV formulation, it is instructive to illustrate the FV discretization (Fig. 2) of Eq. (1) by integrating it over time and space. The 5 term on the right-hand side of Eq. (1) is evaluated using operator splitting, and therefore, the integration here only covers the viscous part as follows:

\[
\begin{align*}
U_{ij}^{n+\Delta t} &= U_{ij} + \frac{1}{\Delta t} \int_0^{\Delta t} \left( F_{i+1/2}(t,x) - F_{i-1/2}(t,x) \right) dx dt \\
&\quad + \frac{1}{\Delta r} \int_0^{\Delta r} \left( G_{j+1/2}(t,r) - G_{j-1/2}(t,r) \right) dr dt
\end{align*}
\]

where:

\[
\begin{align*}
U_{ij} &= U(r_i, x_j), \\
U_{ij}^{n} &= \frac{1}{\Delta t} \int_{t_{i-1/2}}^{t_{i+1/2}} \int_{x_{j-1/2}}^{x_{j+1/2}} U_{ij} \, dx \, dr \\
F_{i+1/2}(t,x) &= F \left( t, r_i + \frac{1}{2} \Delta r, x \right), \\
G_{j+1/2}(t,x) &= G \left( t, r_j + \frac{1}{2} \Delta x, x \right), \\
\Omega_{ij} &= \left[ \left( \frac{r_j - r_{i+1/2}}{2} \right) \times \left( \frac{x_j - x_{i+1/2}}{2} \right) \right], i = 1 \ldots N_r, j = 1 \ldots N_\theta
\end{align*}
\]

and \( \Delta r \) and \( \Delta x \) are the radial, axial and time discretization step sizes, respectively. Equation (6) shows that, at each time level \( t + \Delta t \), at least four fluxes need to be computed, namely \( F_{i-1/2}, F_{i+1/2}, G_{j-1/2} \) and \( G_{j+1/2} \) at the corresponding cell interfaces \((r_i - \frac{1}{2} \Delta r, x), (r_i + \frac{1}{2} \Delta r, x), (r_j - \frac{1}{2} \Delta x, x) \) and \((r_j + \frac{1}{2} \Delta x, x)\), respectively. The solution of the Riemann problem is used to evaluate those fluxes (Fig. 3) at the cell interfaces.
3.1 Riemann solution at the cell interface

The Riemann solution at the cell interface is:

\[
\begin{align*}
\rho^* &= \frac{((V_{t}^l + a)^2 - (\rho V_{t}^R)^2) - ((V_{t}^l - a)^2 - (\rho V_{t}^L)^2)}{2a} \\
\rho V_{t}^* &= \frac{((V_{t}^l + a)^2 - (\rho V_{t}^R)^2)(V_{t}^l - a) - ((V_{t}^l - a)^2 - (\rho V_{t}^L)^2)(V_{t}^l + a)}{2a} \\
\end{align*}
\]  

(8)

and

\[
\begin{align*}
\rho^* &= \frac{((V_{s}^l + a)^2 - (\rho V_{s}^R)^2) - ((V_{s}^l - a)^2 - (\rho V_{s}^L)^2)}{2a} \\
\rho V_{s}^* &= \frac{((V_{s}^l + a)^2 - (\rho V_{s}^R)^2)(V_{s}^l - a) - ((V_{s}^l - a)^2 - (\rho V_{s}^L)^2)(V_{s}^l + a)}{2a} \\
\end{align*}
\]  

(9)

where the * indicates variables at the cell interface and the superscripts L and R indicate the variables at the left and right sides of the cell interface, respectively (Fig. 4). The superscript \( t \) denotes explicit variables (i.e. variables evaluated at the earlier time step). The details of the Riemann solution are given in Appendix A.

3.2 Order of accuracy

To achieve second-order accuracy in space, linear reconstruction within each cell is required (Fig. 5). Since a directional splitting operator is used for the Riemann solution (Eqs (A1) and (A2)), the sub-cell reconstruction is also split in the radial and axial directions. The MUSCL-Hancock approach is adopted to obtain second-order accuracy in space and time.

Similarly, using a directional splitting operator, third-order accuracy is achieved through one-dimensional quadrature reconstruction within each control volume. A WENO approach is adopted to obtain third- and fifth-order accuracy in space.

3.3 Evolution stage of numerical fluxes along the cell interface

This section discusses the evolution stage of the numerical flux along the cell interface. As shown in Eq. (6), to update the variables at the next time level \( t + \Delta t \), two integrals (for an explicit scheme) must be evaluated that describe the distribution of fluxes along the cell interface (Fig. 6). Two cases are discussed below depending on whether the system (Eq. (1)) is linear or not.

3.3.1 Linear scheme for linear flow applications

Water hammer flows are weakly nonlinear. For most water hammer applications, convective terms usually responsible for generating instabilities are typically neglected. This is because water hammer flows have very low Mach numbers (\( M = V^*/a \)) where \( a \) is the wave speed and \( V^* \) is a characteristic flow velocity (e.g. \( V_x \) or \( V_r \)). The typical wave speed in metallic and concrete water-filled conduits is about 1000 m s\(^{-1}\) whilst the flow velocity is usually around 1 m s\(^{-1}\), resulting in Mach numbers of the order 10\(^{-3}\). Accordingly, this work considers only linear systems in which space integrals in Eq. (6) are easily
3.4 Time evolution

The MUSCL-Hancock approach used for the second-order scheme provides second-order accuracy in space and time. However, for third- and fifth-order schemes, a RK method is used for time evolution. Jiang and Shu (1996) emphasize that high order schemes should use total variation diminishing (TVD) RK methods to ensure stability of the scheme. However, it is found (but not shown here) that the third-order TVD RK scheme produces strong spurious oscillations. The second-order TVD RK scheme gives stable results; however, in this work classical explicit RK method is used because it is generally used in the literature and produces stable results at second, third and fourth order. Details of the two- and three-stage TVD RK method are discussed in Jiang and Shu (1996), and the details of the classical RK method can be found in Hirsch (1988b).

3.5 Viscous terms

When viscous flow is considered, the viscous terms (Eq. (3)) are discretized using a second-order central finite difference scheme. Then, operator splitting is used to compute the following equation:

$$\frac{d\overline{U}}{dt} = S$$

(11)

where $\overline{U}$ is given by Eq. (A2). The classical explicit RK method (Hirsch, 1988b) is used to solve Eq. (11).

3.6 Boundary conditions

Boundary conditions are imposed at four different locations namely: at the pipe wall ($r = R$), at the pipe centreline ($r = 0$), at the upstream boundary of the pipe ($x = 0$) and at the downstream boundary of the pipe ($x = L$), where $R$ and $L$ are the pipe radius and length, respectively. Details of the boundary conditions used are listed below.

3.6.1 Reflective boundary conditions

Reflective boundary conditions are boundaries where the impedance is zero. This means that at such boundaries, waves (or flow) are totally reflected (e.g. a wall). Reflective boundaries are imposed at the pipe wall and are given by:

$$\begin{align*}
\frac{\partial P}{\partial r} \bigg|_{r=R} &= 0 \\
V_r \bigg|_{r=R} &= 0 \\
\frac{\partial V_x}{\partial r} \bigg|_{r=R} &= 0 \text{ if } n_r = 0 \\
V_z \bigg|_{r=R} &= 0 \text{ if } n_r = 1
\end{align*}$$

(12)

$$\begin{align*}
P(r_N, -N_x + n_y, x_j) &= P(r_N, -N_x + n_y + 1, x_j) \\
V_r(r_N, -N_x + n_y, x_j) &= -V_r(r_N, -N_x + n_y + 1, x_j) \\
V_x(r_N, -N_x + n_y, x_j) &= (-1)^{n_y} V_x(r_N, -N_x + n_y + 1, x_j)
\end{align*}$$

Figure 6 Sketch of a flux function at cell interface for a three dimensional application showing the need for integration along the cell interface.
where \( N_{bc} \) is the number of fictitious cells added (equal to 2 for linear reconstruction and 3 for quadratic reconstruction); \( n_\infty = 1, 2, \ldots, N_{bc} \) is a counter; and \( n_i = 0 \) or 1, depending on whether the flow is inviscid or viscous, respectively.

### 3.6.2 Symmetric boundary condition

This type of boundary condition is applied at the pipe centreline to satisfy the axisymmetric behaviour of the flow and is defined as follows:

\[
\begin{align*}
\left. \frac{\partial P}{\partial r} \right|_{r=0} &= 0, \\
\left. V_x \right|_{r=0} &= 0, \\
\left. \frac{\partial P}{\partial r} \right|_{r=0} &= 0
\end{align*}
\]

\[
\Rightarrow \quad P(r_{N_n-n_\infty+1},x_j) = P(r_{N_n+n_\infty},x_j) \\
V_x(r_{N_n-n_\infty+1},x_j) = -V_x(r_{N_n+n_\infty},x_j) \\
V_x(r_{N_n-n_\infty+1},x_j) = V_x(r_{N_n+n_\infty},x_j)
\]

(13)

### 3.6.3 Non-reflective boundary conditions

Non-reflective boundary conditions are boundaries where the impedance does not change (meaning that waves, or flow, continue to propagate without any reflection). The non-reflective boundary conditions are usually used to simplify numerical test cases. For example, when the domain of application is too large (e.g., a long river) or unbounded (e.g., ocean, atmosphere), numerical simulation becomes very expensive. Therefore, non-reflective boundary conditions are introduced to shorten the size of the numerical domain. In the current work, to eliminate unwanted boundary reflections and study solely the behaviour of the high frequency waves in a pressurized water-filled pipe, it is necessary to consider an unbounded pipe system. An infinitely (long) pipe is computationally very expensive to model. Instead, non-reflective boundary conditions at the upstream and downstream pipe boundaries are considered. Different non-reflective boundary conditions are available in the literature, including the characteristic boundary condition (CBC) (Hirsch, 2007; Thompson, 1987, 1990), buffer zone techniques and perfectly matched layer (PML) (Richards, Zhang, Chen, & Nelson, 2004). All non-reflective boundary conditions induce numerical reflections, and depending on the flow application, some methods perform better than the others (Colonius, 2004; Richards et al., 2004). In general, all non-reflective boundaries are affected by the following two major factors:

1. **Angle of propagation**: waves propagating in the normal (axial) direction towards the boundary induce the least numerical reflection. As the propagation angle deviates from the normal direction, numerical reflections increase. This is the most severe factor affecting non-reflecting boundary conditions (Richards et al., 2004).
2. **Wave frequency**: higher frequencies induce fewer numerical reflections (Gill, Fattah, & Zhang, 2015).

Although not necessarily the most efficient method, the CBC is the most widely used method in dynamic flow applications.

The method uses the Riemann invariant (or characteristic lines) to eliminate waves propagating from the boundary into the numerical domain and is very easy to implement. This work uses the CBC method, and this section gives details of its implementation in the developed schemes. The performance of CBC interaction with high frequency waves is studied in a later section, where the general factors affecting all non-reflecting boundary conditions mentioned in (i) and (ii) above are discussed.

The CBC method is implemented at the upstream \((x = 0)\) and downstream boundaries \((x = L)\) of the pipe, and therefore only the axial direction need be considered. The split system of equations along the axial direction (Eq. (A2)) is rewritten using characteristic variables (Appendix E). Equation (A33) is applied at the upstream and downstream pipe boundaries (Fig. 7) to obtain the unknown variables at the ghost cells as follows:

- **Non-reflective boundary conditions at the upstream boundary**:

\[
\begin{align*}
V_x(r_i,x_{N_n-n_\infty+1}) &= V_x|_{x=\infty} \\
\rho(r_i,x_{N_n-n_\infty+1}) &= \exp \left( \frac{C^+ - C^-}{2a} \right) \\
C^+ &= V_x|_{x=\infty} + a \log(\rho|_{x=\infty}) \\
C^- &= V_x(r_i,x_{N_n-n_\infty+2}) - a \log(\rho(r_i,x_{N_n-n_\infty+2}))
\end{align*}
\]

where the subscript \( \infty \) refers to values at infinity (i.e., values at the boundary if the pipe was infinitely long).

- **Non-reflective boundary conditions at the downstream boundary**:

\[
\begin{align*}
V_x(r_i,x_{N_n-N_n+n_\infty}) &= V_x|_{x=\infty} \\
\rho(r_i,x_{N_n-N_n+n_\infty}) &= \exp \left( \frac{C^+ - C^-}{2a} \right) \\
C^+ &= V_x(r_i,x_{N_n-N_n+n_\infty}) + a \log(\rho(r_i,x_{N_n-N_n+n_\infty})) \\
C^- &= V_x|_{x=\infty} - a \log(\rho|_{x=\infty})
\end{align*}
\]

(15)
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3.6.4 Source boundary condition

The numerical test set-up used in this work for source boundary condition is depicted in Fig. 8, where a wave is generated from a source located at \( x = L \) and only the left going waves are considered. The wave form at the source is given in Fig. 9. This form is chosen because it is smooth and allows the modeller to select the desired frequency bandwidth (FBW). Its mathematical form is:

\[
P_F(t) = P_s \exp \left( -4 \frac{w_c^2}{\beta^2} \log(10) \left( t - \frac{\beta}{w_c} \right)^2 \right) \sin \left( w_c \left( t - \frac{\beta}{w_c} \right) \right); \quad \text{where } 0 < t \leq t_{\text{wave}} = \frac{\beta}{w_c} \quad (16)
\]

where \( w_c = 2\pi f_c \) is the angular central frequency (in rad s\(^{-1}\)) with \( f_c \) the central frequency (in Hz); \( P_F \) is the pressure at the source; \( P_s = 0.1P_0 \) is the maximum pressure induced by the wave at the source with \( P_0 \) the initial pressure in the pipe; \( t_{\text{wave}} \) is the duration of the wave generated at the source; and \( \beta \) is a coefficient that controls the FBW. The effect of \( \beta \) is shown in Figs 9 and 10 which show the cases for \( \beta = 16\pi \) and \( \beta = 80\pi \), respectively. The source is considered to be circular in shape with a given diameter \( D_s \) and is located at the pipe centreline (Fig. 8). Initially at time \( t = 0 \) s, the fluid is at rest. The boundary conditions at the source are given by:

\[
\begin{align*}
V_r(r, x_{N_c - N_{bc} + n_j}) &= 0 \\
P(r, x_{N_c - N_{bc} + n_j}) &= P_0 + P_F \\
\frac{\partial V}{\partial x} \bigg|_{x=0} &= 0 \quad (17)
\end{align*}
\]

3.6.5 Boundary conditions for reservoir-pipe-valve system

A reservoir-pipe-valve (RPV) system (Fig. 11) is considered to study the generation of radial waves due to a rapid valve closure. This test is known as the classical water hammer test case. The boundary conditions for a reservoir located at the upstream boundary are given by:

\[
P \big|_{x=0} = P_{\text{res}} \quad ; \quad \frac{\partial V}{\partial x} \bigg|_{x=0} = 0 \quad \text{and} \quad \frac{\partial V}{\partial x} \bigg|_{x=0} = 0 \quad (18)
\]

where \( P_{\text{res}} \) is the fixed pressure at the reservoir. The boundary conditions for a valve located at the downstream boundary are given by:

\[
\frac{\partial P}{\partial x} \bigg|_{x=L} = 0 \quad ; \quad V_r \big|_{x=L} = 0 \quad ; \quad V_x \big|_{x=L} = 0 \quad (19)
\]

Note that in the case of sudden valve closure (non-smooth wave), the third/fifth-order WENO scheme (Appendix C) would achieve at most third-order accuracy. The robustness and accuracy of the scheme for simulating sharp (non-smooth) waves are discussed later.
4 Features of the computational schemes

In this section, the test cases use the pipe setup in Fig. 8 where the flow is initially at rest. A smooth transient wave is generated at the source \((x = L)\) with waveform as shown in Fig. 10. Only inviscid flow is considered to study the performance of Riemann solver schemes. In all test cases, the radial and axial step sizes are equal \((\Delta r = \Delta x)\), where \(\Delta r = R/N_r; \Delta x = L/N_r\) to obtain the same mesh-related dissipation along the different flow directions.

4.1 Stability and accuracy order of the scheme

Stability and order of accuracy are determined using the mesh refinement technique. The scheme shows stability for Courant–Friedrich–Lewy (CFL) number at about \(a\Delta t/\Delta x = 0.1\) to 0.3. For consistency in this work, a value of CFL = 0.1 is used for all test cases. The \(l^2\) and \(l^1\) norms (e.g. Chaudhry & Hussaini, 1985; Roy, 2003) are used to compute the accuracy of the scheme. The results are summarized in Tables 1, 2 and 3 for the second-, third- and fifth-order schemes, respectively. Tables 1, 2 and 3 show that the orders of accuracy of these three schemes are about 1.8, 2.9 and 4.5, respectively. The procedure of computing the order of accuracy is as follows:

\[
\text{Order} = \frac{\log(\chi/\chi^{ref})}{\log(N_r^{ref}/N_r)} \quad \chi \equiv \{l^1, l^2\} \tag{20}
\]

with

\[
\begin{align*}
\ell^1 & = \sum_{N_r} \frac{P - P_{\text{converged}}}{\chi^{ref}} \\
\ell^2 & = \sqrt{\sum_{N_r} (P - P_{\text{converged}})^2}
\end{align*} \tag{21}
\]

Table 1 Mesh refinement convergence tests for the second-order scheme \((f_c = 1000 \text{ Hz})\)

<table>
<thead>
<tr>
<th>#</th>
<th>Order ((l^2)) norm</th>
<th>Order ((l^1)) norm</th>
</tr>
</thead>
<tbody>
<tr>
<td>(N_r = 5)</td>
<td>- ref -</td>
<td>- ref -</td>
</tr>
<tr>
<td>(N_r = 10)</td>
<td>1.2016</td>
<td>1.1663</td>
</tr>
<tr>
<td>(N_r = 20)</td>
<td>1.3529</td>
<td>1.3813</td>
</tr>
<tr>
<td>(N_r = 30)</td>
<td>1.5010</td>
<td>1.5307</td>
</tr>
<tr>
<td>(N_r = 40)</td>
<td>1.7836</td>
<td>1.8050</td>
</tr>
<tr>
<td>(N_r = 50)</td>
<td>converged</td>
<td>converged</td>
</tr>
</tbody>
</table>

Table 2 Mesh refinement convergence tests for third-order scheme \((f_c = 500 \text{ Hz})\)

<table>
<thead>
<tr>
<th>#</th>
<th>Order ((l^2)) norm</th>
<th>Order ((l^1)) norm</th>
</tr>
</thead>
<tbody>
<tr>
<td>(N_r = 3)</td>
<td>1.9438</td>
<td>1.9369</td>
</tr>
<tr>
<td>(N_r = 5)</td>
<td>2.0686</td>
<td>2.0395</td>
</tr>
<tr>
<td>(N_r = 10)</td>
<td>2.3198</td>
<td>2.2771</td>
</tr>
<tr>
<td>(N_r = 15)</td>
<td>2.5913</td>
<td>2.5605</td>
</tr>
<tr>
<td>(N_r = 20)</td>
<td>2.9467</td>
<td>2.9284</td>
</tr>
<tr>
<td>(N_r = 30)</td>
<td>- ref -</td>
<td>- ref -</td>
</tr>
<tr>
<td>(N_r = 40)</td>
<td>converged</td>
<td>converged</td>
</tr>
</tbody>
</table>

Table 3 Mesh refinement convergence tests for fifth-order scheme \((f_c = 1000 \text{ Hz})\)

<table>
<thead>
<tr>
<th>#</th>
<th>Order ((l^2)) norm</th>
<th>Order ((l^1)) norm</th>
</tr>
</thead>
<tbody>
<tr>
<td>(N_r = 25)</td>
<td>- ref -</td>
<td>- ref -</td>
</tr>
<tr>
<td>(N_r = 50)</td>
<td>2.9599</td>
<td>3.3352</td>
</tr>
<tr>
<td>(N_r = 100)</td>
<td>5.5955</td>
<td>5.8355</td>
</tr>
<tr>
<td>(N_r = 200)</td>
<td>4.5717</td>
<td>4.8484</td>
</tr>
<tr>
<td>(N_r = 400)</td>
<td>converged</td>
<td>converged</td>
</tr>
</tbody>
</table>

where the superscript ref and subscript converged refer respectively to the reference and converged cases which are shown in Tables 1, 2 and 3.

A different approach based on computing the energy flux rather than the \(l^1\) and \(l^2\) norms could be used to determine the order of accuracy of a given scheme. For this purpose, the dissipation rate is defined as the rate of decay of energy flux (defined in Eq. (23)) measured 10 m from the source location. The period over which the energy flux is integrated is chosen such that the total injected energy passes through the cross-sectional area 10 m from the source. The dissipation rate is defined as follows:

\[
(\% \text{Numerical dissipation per 10 m}) = \left[ 1 - \frac{E_F(x = L - 10)}{E_F(x = L)} \right] \times 100 \tag{22}
\]

Table 4 Mesh refinement convergence tests for the second-order scheme based on energy flux \((f_c = 1000 \text{ Hz})\)

<table>
<thead>
<tr>
<th>#</th>
<th>Order ((\chi \equiv E_F))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(N_r = 5)</td>
<td>- ref -</td>
</tr>
<tr>
<td>(N_r = 10)</td>
<td>1.0126</td>
</tr>
<tr>
<td>(N_r = 20)</td>
<td>1.7741</td>
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<tr>
<td>(N_r = 40)</td>
<td>1.8284</td>
</tr>
<tr>
<td>(N_r = 50)</td>
<td>1.8484</td>
</tr>
</tbody>
</table>

Table 5 Mesh refinement convergence tests for third-order scheme based on energy flux \((f_c = 1000 \text{ Hz})\)

<table>
<thead>
<tr>
<th>#</th>
<th>Order ((\chi \equiv E_F))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(N_r = 5)</td>
<td>- ref -</td>
</tr>
<tr>
<td>(N_r = 10)</td>
<td>2.4007</td>
</tr>
<tr>
<td>(N_r = 15)</td>
<td>2.8725</td>
</tr>
<tr>
<td>(N_r = 20)</td>
<td>2.9305</td>
</tr>
</tbody>
</table>

Table 6 Mesh refinement convergence tests for fifth-order scheme based on energy flux \((f_c = 4000 \text{ Hz})\)

<table>
<thead>
<tr>
<th>#</th>
<th>Order ((\chi \equiv E_F))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(N_r = 10)</td>
<td>- ref -</td>
</tr>
<tr>
<td>(N_r = 20)</td>
<td>4.4993</td>
</tr>
<tr>
<td>(N_r = 30)</td>
<td>4.8227</td>
</tr>
</tbody>
</table>
with \( E_F = \int_0^{2\pi} \int_0^R \int_0^{t_{\text{ keep}}} [P(x = L - 10) \times V_x(x = L - 10)] r dr d\theta dt \) \( (23) \)

where \( t_{\text{ keep}} \) is a measurement time sufficiently long to allow all waves to propagate through the cross section 10 m from the source. Using the dissipation rate defined in Eq. (22) as the norm \( \chi \) in Eq. (20), the computed accuracy of the second-, third- and fifth-order schemes given in Tables 4, 5 and 6 shows that the order of these schemes are 1.84, 2.9 and 4.8, respectively, which is consistent with the results from \( l^1 \) and \( l^2 \) norms.

### 4.2 Dissipation and computational time

In this section, the dissipation rate defined in Eq. (22) is examined. It is observed that, for energy dissipation below 2\%, the measured pressure decay at a given location becomes low. Therefore, 2\% energy dissipation is set as a minimum convergence target for the different schemes.

The computational (CPU) time is assessed based on a non-parallel and non-optimized algorithm. For consistency, all test cases are run under the same algorithm structure. Therefore, the CPU time computed here is a basic computing scenario and could be improved using optimized/parallelized algorithm structures. In this study, the CPU times are given in minutes.

Figure 12 compares the dissipation rate and CPU time variation with \( N_r \) between the second- and third-order schemes for the case of \( f_c = 500 \) Hz. Figure 12a shows that the dissipation rate is similar for these two schemes at a frequency of 500 Hz. For \( N_r = 10 \) and \( N_r = 15 \), the dissipation rate of the third-order scheme is slightly less than the second order, but both are less than 2\%. However, Fig. 12b shows that the CPU time for the third-order scheme is twice as long as that for the second-order scheme. This makes the second-order scheme more efficient for simulating very low frequency (\( f < 500 \) Hz) waves.

Figure 13 compares the dissipation rate and the CPU time between the second-, third- and fifth-order schemes for the case of \( f_c = 1000 \) Hz. Figure 13a shows that the second-, third- and fifth-order schemes achieve dissipation rates less than 2\% for \( N_r = 40, N_r = 20 \) and \( N_r = 5 \), respectively. In comparison with \( f_c = 500 \) Hz case (Fig. 12), Fig. 13b shows that the third-order scheme becomes much more efficient than the second-order scheme for simulating waves at a frequency of 1 kHz. Figure 13 shows that, for \( f_c = 1000 \) Hz, the fifth-order scheme gives slightly better efficiency compared with the third-order scheme. On the other hand, Fig. 14 gives the case
for a higher frequency of $f_c = 4000$ Hz and shows that the efficiency of the fifth-order scheme increases whereas the second- and third-order schemes are no longer computationally practical. In fact, Fig. 14 shows that the fifth-order scheme converges to 2% dissipation for $N_r = 20$, while the third-order scheme converges to 50% after five times the third-order CPU time (Fig. 14b) at $N_r = 40$. The second-order scheme has an even lower efficiency than the third-order scheme. This shows that the second- and third-order schemes become computationally very expensive and are unsuitable for simulating wave propagation at frequencies higher than 1 kHz.

Figures 12, 13 and 14 illustrate a need for higher order schemes as the wave frequency increases. It is therefore important to consider which order of accuracy is most suitable for simulating wave propagation at a given frequency. The rest of this section develops an empirical equation that gives an approximate “most efficient order of accuracy” for a given frequency and desired CPU time based on the current RS (Riemann solver) schemes.

4.2.1 Approximate “most efficient order of accuracy”

$N_S$ is the number of discretized FV per central wavelength ($a/f_c$) and is given by the following relation:

$$N_S = \frac{a}{f_c \Delta r} = \frac{a}{f_c R} = \frac{a}{f_c L}$$

(24)

Figure 15 compares dissipation rate with respect to $N_S$ for the second-, third- and fifth-order schemes at different central frequencies ($f_c$). Figure 15 shows that each scheme converges for a specific $N_S$ value for different frequencies. For example, the second-, third- and fifth-order schemes converge respectively for an averaged $N_S^b$ close to $N_S^{2}_S = 250$, $N_S^{3}_S = 130$ and $N_S^{5}_S = 25$ where the superscript $b$ denotes the scheme order. Across a range of parameter space, numerical tests and fitting, it is found

![Figure 14 Comparison of the dissipation rate and CPU time between the second-, third- and fifth-order schemes for the case of $f_c = 4000$ Hz: (a) dissipation rate; (b) CPU time](image1)

![Figure 15 Comparison of the dissipation rate with change of number of discretized FV per wavelength between the second-, third- and fifth-order schemes](image2)

![Figure 16 Approximate “most efficient order of accuracy” (Eq. (31)) variation with central frequency ($f_c$)](image3)
that the following empirical equation applies:

$$\frac{N_b^b}{N_b^S} = \left( \frac{b}{b_0} \right) \left( \frac{R}{R_0} \right)^{\frac{1}{\gamma_b f_c}}$$  \hspace{1cm} (25)

where $b_0$ is a reference scheme order of accuracy. For example, if $b_0 = 2$ and $b = 5$, Eq. (25) gives $N_S^2/N_S^5 \approx 10$ which is close to the $N_S$ ratio ($\sim 250/25$) between the fifth- and second-order schemes observed in Fig. 15.

Moreover, at convergence (dissipation rate $< 2\%$), Eq. (24) gives:

$$N_r^b = N_S^b \frac{R f_c}{\Gamma_b}$$  \hspace{1cm} (26)

where $N_r^b$ is the $N_r$ used for the scheme of order $b$ at convergence. On the other hand, the CPU time variation with $N_r$ for different schemes is fitted exponentially as follows:

$$T_{CPU}^b = \Gamma_b \exp(\gamma_b N_r)$$  \hspace{1cm} (27)

where $T_{CPU}^b$ is the fitted CPU time function for a given scheme of order $b$; $\Gamma_b$ is a constant defining the CPU time at very low $N_r$; and $\gamma_b$ is the rate at which the CPU time increases. At convergence, Eq. (27) gives:

$$N_r^b = \frac{1}{\gamma_b} \log \left( \frac{T_{CPU}^b}{\Gamma_b} \right)$$  \hspace{1cm} (28)

Combining Eq. (26) and Eq. (28) yields:

$$N_r^b = \frac{a}{\gamma_b f_c} \log \left( \frac{T_{CPU}^b}{\Gamma_b} \right)$$  \hspace{1cm} (29)

Inserting Eq. (29) into Eq. (25) yields:

$$\left( \frac{b}{b_0} \right) \left( \frac{R}{R_0} \right)^{\frac{1}{\gamma_b f_c}} = \frac{N_r^b}{N_r^S} \frac{R}{\log(T_{CPU}^b/\Gamma_b)}$$  \hspace{1cm} (30)

It is found that the rate $\gamma_b$ is similar for the three different schemes and is about 0.084, 0.089 and 0.092 for the second-, third- and fifth-order schemes, respectively. To a reasonable approximation, $\gamma_b$ may be assumed to be fixed at 0.1. For simplicity, $\Gamma_b$ is also fixed to an average value of 1 min per 10 m wave propagation for all schemes when using small $N_r$, and when $\Gamma_b$ increases linearly with the propagation length. For example, for 20 m propagation length, $\Gamma_b$ becomes 2 min, and so on. Choosing the second order as a reference scheme order...
(b₀ = 2) and assuming that N₂ is known, Eq. (30) for 10 m wave propagation length becomes:

\[ \left( \frac{b}{2} \right)^{\frac{1}{2}} = \frac{N₂}{\log(T_{PU}^a)} \frac{R}{a} = 0 \]  \hspace{1cm} (31)

Applying Eq. (31) for the case of R = 0.2 m, a = 1000 m s⁻¹, N₂ = 250 and setting T_{PU} to be 15 min, the variation of the scheme’s order of accuracy with f_c could be obtained as shown in Fig. 16. Figure 16 shows that for frequencies from 5 kHz to 100 kHz, the required order of accuracy increases from 5 to 8.

4.3 Performance of CBC to model non-reflective boundary condition

Numerical reflections from non-reflective boundaries are difficult to eliminate and depend significantly on the propagation angle of the wave (Gill et al., 2015). Louati and Ghidaoui (2017a, 2017b) show that radial wave modes propagate at different angles and these angles vary with frequency (Figs 17 and 18). This section studies the performance of CBC for high frequency wave simulation. The numerical reflections are studied in terms of energy computation using the fifth-order scheme. Let E_T denote the total injected wave energy, E_n the energy content in a given mode n, and E_{Rn} the reflected energy from the boundary for a given mode n. The energy quantities are computed using the numerical data measurements. The plane mode reflections are studied by considering a source diameter (Fig. 8) D_s = D such that no radial modes are generated. This case is referred to as case 0. When studying the high mode cases, the source diameter is D_s = 0.2 D. The frequency bandwidth is [0.9 f_c to 1.1 f_c] where f_c = 3720 Hz and f_c = 6800 Hz cases are both used. The f_c = 3720 Hz case excites only the plane mode (M0) and the first high mode (M1). This case is referred to as case 1. The f_c = 6800 Hz case excites M0, M1 and the second high mode (M2). This case is referred to as case 2. The numerical reflections for the three different cases are shown in Figs 19 and 20.

Table 7 gives the energy reflection for case 0 for the two different central frequencies (f_c = 3720 Hz and f_c = 6800 Hz). For case 0, the propagation angle is zero, and therefore, the difference in the reflected energy is only due to the change in frequency. Table 7 shows that numerical reflection decreases as the frequency increases. Such a feature is usually observed in other acoustic applications (Gill et al., 2015; Richards et al., 2004). For case 0, the ratio between the energy reflections for f_c = 3720 Hz and f_c = 6800 Hz is about 1.56. This ratio is used as a reference value for change in energy reflection due solely to a change in frequency.

Table 8 gives energy reflection for case 1 and case 2. Table 8 shows that the M1 energy reflection is much higher for case 1 (f_c = 3720 Hz) than for case 2 (f_c = 6800 Hz). The ratio between the M1 energy reflections in case 1 and 2 is about 26. Although the frequency is higher for case 2, the reflection ratio seems too large (26 > 1.56) to be due only to frequency change. This large difference is, in fact, due to the change in propagation angle. As seen from Table 8, the M1 propagation angle for case 1 is about twice the value for case 2. This illustrates that variation in the propagation angle can dramatically affect the numerical reflection. A similar effect is also observed in other acoustic applications using CBC and other non-reflective boundary condition methods (Gill et al., 2015; Richards et al., 2004).

From Table 8, the range of propagation angle is the same for M2 in case 2 and M1 in case 1. Therefore, their energy reflection could be compared based on the change in frequency and mode number only. Table 8 shows that the ratio of energy reflection between M1 and M2 is about 1.4 (< 1.56) which is only slightly lower than the energy reflection ratio due solely to change in frequency. This shows that high modes have almost similar reflection rates at the same propagation angle. The slight difference in reflection ratio between M1 and M2 could be due to the radial wave form of high modes (Louati & Ghidaoui, 2017b).

![Figure 21](image-url) Pressure variation with time at the pipe centreline and at x/L = 0.5 for different ε values (fifth-order scheme; f_c = 1000 Hz; D_s = D): (a) ε = 10⁻¹⁰; (b) ε = 10⁻⁶; (c) ε = 10⁻³; (d) ε = 10⁻⁴
4.4 WENO “calibration” parameter

In the WENO reconstruction procedure (Appendix C), a calibration parameter ($\varepsilon$) (Eq. (A24)) is introduced in the weighting functions to avoid singularities. Note that this is a purely numerical parameter (tuning parameter) used to stabilize the scheme (in other words, it introduces numerical dissipation). More details are given in Jiang and Shu (1996). This section studies the effect of $\varepsilon$ on the numerical results. Figure 21 gives pressure variation with time for different $\varepsilon$ values. Figure 21a shows that for $\varepsilon = 10^{-10}$ the signal has spurious oscillations. However, for $\varepsilon = 10^{-6}$ (Fig. 21b), the spurious oscillations are eliminated. When $\varepsilon$ is further increased to $10^{-5}$ (Fig. 21c), the numerical reflection is reduced and is almost eliminated at $\varepsilon = 10^{-4}$ (Fig. 21d). However, when comparing the case of $\varepsilon = 10^{-4}$ (Fig. 21d) and $\varepsilon = 10^{-6}$ (Fig. 21b), it is found that the pressure amplitude is slightly reduced for the value of $\varepsilon = 10^{-4}$. It is thought that the almost complete elimination of the M0 reflection may be due to slightly increased numerical dissipation as $\varepsilon$ is increased. In this work, $\varepsilon = 10^{-6}$ is used as a lower threshold for this tuning parameter. The best value to use may change depending on the test case. This brief discussion regarding the effect $\varepsilon$ has on numerical reflections is important because the literature usually reports that this parameter should be very small (e.g. $\varepsilon = 10^{-12}$). In fact, when choosing very small values for $\varepsilon$, the user of the scheme may be misled into thinking that the scheme is not accurate.

### Table 8 M1 and M2 wave reflection ($D_s = 0.2 \, D$)

<table>
<thead>
<tr>
<th>Case</th>
<th>Mode</th>
<th>Frequency range (in Hz)</th>
<th>Angle range (°)</th>
<th>$\varepsilon$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>M1</td>
<td>[3350 to 4100] $f_c = 3720$</td>
<td>[48 to 65] $\theta_{k1}(f_c) = 55$</td>
<td>10.57%</td>
</tr>
<tr>
<td>2</td>
<td>M1</td>
<td>[6120 to 7480] $f_c = 6800$</td>
<td>[24 to 30] $\theta_{k1}(f_c) = 26.6$</td>
<td>0.3%</td>
</tr>
<tr>
<td>2</td>
<td>M2</td>
<td>[6120 to 7480] $f_c = 6800$</td>
<td>[48 to 65] $\theta_{k2}(f_c) = 55$</td>
<td>7.54%</td>
</tr>
</tbody>
</table>
5 Classical water-hammer test case: rapid valve closure in a RPV system

5.1 Numerical dispersion (classical water hammer test case)

This section considers a classical test case for water hammer application consisting of a rapid valve closure in a RPV system with initial Poiseuille flow. The pipe system and boundary conditions for this test case are given in Section 3.6.5. This test case was first considered by Mitra and Rouleau (1985) to numerically study radial waves in liquid transmission lines. Usually radial waves are not considered in water hammer models because water hammer waves are generated by mechanical devices such as valves and pumps whose induced wave frequency is far below the cut-off frequency of any higher mode. However, Mitra and Rouleau (1985) consider an ideal sudden valve closure that induces a broad band signal (with large FBW) that excites high modes, and yet they report that radial pressure waves are only large near the valve and decay rapidly with distance from the valve. The reason for this observation is that the numerical scheme used by Mitra and Rouleau (1985) was implicit in time and of low order accuracy, therefore the numerical dissipation is too high for accurately simulating HFW. This work uses the developed high-order explicit scheme to study the radial wave propagation due to sudden valve closure, and does not suffer from this problem.

Rapid valve closure in a RPV system is a widely used illustrative test case for water-hammer applications (Wylie, Streeter, & Suo, 1993). Inviscid and viscous-laminar flows are considered. The initial conditions for the inviscid flow case are uniform radial velocity distribution and constant initial pressure along the pipe, given as follows:

\[
V_x(r) = V_0 \left(1 - \frac{r^2}{R^2}\right) \quad 0 \leq r \leq R \\
P(x) = P_0 = \rho_0 g H_0 \quad 0 \leq x \leq L
\]

where \( v \) is kinematic viscosity in water (\( v \approx 10^{-6} \text{m}^2/\text{s} \)); \( R_e \) is Reynolds number (fixed at \( R_e = 500 \)); \( D \) is pipe diameter (\( D = 0.4 \text{m} \)); \( V_0 \) is constant initial velocity; \( \rho_0 \) is initial density (\( \rho_0 = 1000 \text{kg m}^{-3} \)); \( H_0 \) is initial pressure head (\( H_0 = 10 \text{m} \)); \( P_0 \) is constant initial gauge pressure in the pipe. The viscous-laminar flow case is a Poiseuille flow with details as follows:

\[
V_x(r) = 2V_0 \left(1 - \frac{r^2}{R^2}\right) \quad 0 \leq r \leq R \\
P(x) = P_0 - \frac{8\rho_0 V_0 x}{\pi R^2} \quad 0 \leq x \leq L
\]

For both cases, the length of the pipe is chosen to be \( L = 10 \text{m} \). For the case of inviscid flow (and where nonlinear convective terms are neglected because of very low Mach number), the method of characteristics (MOC) gives an exact solution.
for a sudden valve closure in a RPV system. Figure 22 gives
the dimensionless pressure variation with time measured at
the valve where the pressure magnitude is normalized by the
Joukowsky pressure ($P_{jou} = \rho_0 V_0^2 a$). The exact solution is com-
pared with the results from the second-order MUSCL-Hancock
and the fifth-order WENO schemes in Fig. 22a and 22b, respec-
tively. The results show that both schemes give accurate results
except that the fifth-order WENO scheme induces localized
spurious oscillations near severe pressure jumps (Fig. 22b).

Figure 23a and 23b gives the dimensionless pressure vari-
ation with time for the case of viscous-Poiseuille flow using
the second-order MUSCL-Hancock and the fifth-order WENO
schemes, respectively. In this case, radial waves are excited
at the valve and therefore the oscillations observed in Fig. 23
should not be thought of as spurious oscillations. The fact that
the increase in pressure at the valve is twice as large as the
Joukowsky pressure is because the maximum velocity at the
centreline is twice the average velocity for a steady state pipe
Poiseuille flow (Eq. (33)). These radial waves are discussed
in more detail in the next section. The second-order MUSCL-
Hancock and the fifth-order WENO schemes are compared in
Fig. 24 and the results show that both schemes agree well
except at the sudden jump where local pre/post-jump spurious
oscillations are generated by the fifth-order WENO scheme.
These spurious oscillations are small and are eliminated if the
jump (i.e. valve closure) is slightly less severe. In fact, Fig.

Figure 26 Dimensionless pressure variation at the valve with time: (a)
at the pipe centreline; (b) at the pipe wall

Figure 27 Dimensionless pressure variation at the valve and at the
pipe wall for a period of $L/a$ and its frequency domain transforma-
tion: (a) dimensionless pressure variation with time; (b) frequency domain

Figure 28 Dimensionless pressure distribution in the $r-x$ space plane
for the case of sudden valve closure in a RPV system with initial
Poiseuille flow at different times: (a) at time $t = 0.2 L/a$; (b) at time
$t = 0.4 L/a$; (c) at time $t = 0.6 L/a$; (d) at time $t = 0.8 L/a$
this test case, the fifth-order WENO scheme uses a mesh with \( N_r = 10 \) and \( N_x = 500 \) while the second-order MUSCL-Hancock scheme uses \( N_r = 50 \) and \( N_x = 2500 \), and yet the second-order scheme exhibits significantly more dissipation, as shown in Fig. 23a where the radial waves become undetectable after \( t = 4L/a \).

5.2 Discussion of radial waves

In this section, only results obtained from the fifth-order WENO scheme are considered. Figure 26a and 26b give the dimensionless pressure variations with time at the valve measured at the pipe centreline and at the pipe wall, respectively. Figure 26 shows that the magnitude of the pressure oscillations changes with radial position. Figure 27a gives the pressure signal at the pipe wall in Fig. 26b up to \( t = L/a \), and Fig. 27b gives its frequency domain. Figure 27b shows that the pressure oscillations observed in Fig. 26 are due to radial waves excited by the sudden valve closure, where \( f_n (n = 1, \ldots, 5) \) in Fig. 27b are the cut-off frequencies of the first five high modes. These cut-off frequencies are defined by \( f_n = (\alpha_{rn}/\pi)(a/D) \) \((n = 1, \ldots, 5)\), where \( \alpha_{rn} \) is the \( n \)th zero of Bessel function of first kind at order 1 \((J_1)\), which results from imposing the no flux condition at the pipe wall and gives \( \alpha_{rn} = 3.8317, 7.01556, \ldots \), etc. (Louati & Ghidaoui, 2017a, 2017b; Rienstra & Hirschberg, 2003).

Figure 28 gives the dimensionless pressure distribution in r-x plane for different times and shows the propagation of the plane wave mode (M0) and the first radial mode (M1) along the pipe. Although higher modes \((n > 1)\) are excited (Fig. 27b), M1 is dominant and carries most of the energy as shown in Fig. 27b. As discussed in Louati and Ghidaoui (2017a, 2017b), the group velocity of high modes \((n > 0)\) varies with the frequency. This is why Fig. 28 shows that some waves at M1 travel slowly and get spread near the valve, while other waves travel much faster with little spreading (right behind the plane wave mode).

The main point in this section is that the excited radial modes propagate along the pipe if the valve closure time (transient time) is fast enough that the induced frequency is higher than the first cut-off frequency. However, this point was not highlighted by Mitra and Rouleau (1985) only because their scheme was too dissipative to observe these propagations. In fact Fig. 23 shows that these propagations die out when using the second-order scheme, which is more dissipative than the third/fifth-order scheme. This underscores the central point of this paper that characteristics accuracy and dissipation are important when modelling higher wave modes.

6 Conclusions

This paper explores aspects of the 2D Riemann solver-based finite volume numerical scheme developed for appropriately modelling higher mode wave propagation in pipelines. Its robustness and accuracy are tested for modelling dispersive high frequency waves in pipe flows. Different order of accuracy (second, third and fifth) are developed and tested. An empirical equation based on numerical experiments is formulated to provide an approximate order of accuracy needed to simulate HFW for a given central frequency. The fifth-order scheme is found to be accurate for modelling smooth waves up to 10 kHz. However, when simulating test cases with severe jumps, the fifth-order scheme accuracy drops to third order with localized spurious oscillation at the jump locations. It is also shown that the tuning parameter of WENO scheme could have a large influence on the stability of numerical results, and its parameters should be carefully selected for different test cases.

The classical water hammer test case with laminar flow is studied using the developed 2D scheme. It is shown that fast valve closure excites radial modes, and that these radial modes could propagate along the pipe and not only near the boundaries as first discussed by Mitra and Rouleau (1985). Although later research work after Mitra and Rouleau (1985) (e.g. Martins et al., 2016; Martins, Brunone, Meniconi, Ramos, & Covas, 2018; Wang et al., 2016) studied numerically two dimensional water hammer application using more accurate scheme, yet their results do not catch the presence of radial waves. The reason is that radial modes are excited at high frequency and second-order scheme cannot resolve such details unless extremely fine mesh is used; which is tremendously time consuming. This paper shows how a high-order scheme gives a much better performance (optimal CPU time and mesh size) to catch the dispersion details of transient waves at high frequency. It is important to note (at least in transient field) that transient waves generated due to sudden valve closure propagate “in principle” at infinitely wide band frequency. This means that high modes are expected to be excited due to such manoeuvre, and their nonexistence in numerical simulation indicates that the scheme is not accurate enough. In this case, unless high modes are not primary objective of the simulation outcome, higher-order schemes must be used.

Finally, the performance of the characteristic boundary condition (CBC) is also studied for different propagation angles and frequencies. It is found that the wider the propagation angle with respect to the normal (axial) direction, the larger are the numerical reflections. Therefore, in unbounded numerical domain, it is better to avoid simulating HFW propagation at or near the cut-off frequency of high modes since such waves have very large angle of propagation.

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Appendix

A. Riemann solution at the cell interface

The Riemann solution is split along each direction according to the following procedure:

\[
\frac{\partial U_i^{ID}}{\partial t} + \frac{\partial F_i^{ID}}{\partial x} = 0 \quad \Rightarrow \quad U_F \quad \text{along the radial direction} \tag{A1}
\]

and

\[
\frac{\partial U_i^{ID}}{\partial t} + \frac{\partial G_i^{ID}}{\partial x} = 0 \quad \Rightarrow \quad U \quad \text{along the axial direction} \tag{A2}
\]

where

\[
U_r^{ID} = \begin{bmatrix} \rho \\ \rho V_r \\ \rho V_r^2 + P \end{bmatrix}; \quad U_a^{ID} = \begin{bmatrix} \rho \\ \rho V_a \\ \rho V_a^2 + P \end{bmatrix}
\]

\[
F_r^{ID} = \begin{bmatrix} \rho V_r \\ \rho V_r^2 + P \\ \rho V_r \end{bmatrix}; \quad G_r^{ID} = \begin{bmatrix} \rho V_r \\ \rho V_r^2 + P \\ \rho V_r \end{bmatrix}
\]

The Jacobian matrix from Eqs (A1) and (A2) are:

\[
J_F = \frac{\partial F_r^{ID}}{\partial U_r^{ID}} = \begin{bmatrix} 0 & 0 & 1 \\ \rho a^2 - V_r^2 & 2V_r & 0 \\ \rho a^2 - V_r^2 & 2V_r & 0 \end{bmatrix}
\]

\[
J_G = \frac{\partial G_r^{ID}}{\partial U_r^{ID}} = \begin{bmatrix} 0 & 0 & 1 \\ \rho a^2 - V_r^2 & 2V_r & 0 \\ \rho a^2 - V_r^2 & 2V_r & 0 \end{bmatrix}
\]

and their corresponding eigenvalues and right-eigenvectors are:

\[
\lambda_i^F = \begin{cases} \lambda_1^F = V_r - a \\ \lambda_2^F = V_r + a \end{cases}; \quad \lambda_i^G = \begin{cases} \lambda_1^G = V_a - a \\ \lambda_2^G = V_a + a \end{cases}
\]

and:

\[
K_F = \begin{bmatrix} 1 & 0 & 0 \\ V_r - a & 1 & 0 \\ 1 & V_r + a & 1 \end{bmatrix}; \quad K_G = \begin{bmatrix} 1 & 0 & 0 \\ V_a - a & 1 & 0 \\ 1 & V_a + a & 1 \end{bmatrix}
\]

An approximate Riemann solution is obtained by assuming frozen Jacobian matrices at each time step, where explicit variables are used for the Jacobian matrix entries. Therefore, the system is linearized at each time step. Using the generalized Riemann invariant (Toro, 2009) across the characteristic lines (Fig. 4), gives:

\[
\left\{ \begin{array}{l}
\frac{dp}{\tau_r} = \frac{dpV_r}{1}; \quad \text{across } \frac{dx}{dt} = V_r - a \\
\frac{dp}{\tau_r} = \frac{dpV_r}{1}; \quad \text{across } \frac{dx}{dt} = V_r + a
\end{array} \right. \quad \text{along the radial direction} \tag{A7}
\]

which leads to the Riemann solution at the radial and axial cell interfaces respectively as follows (Fig. 4):

\[
\rho* = \frac{\left( (V_r^t + a)\rho^R - (\rho V_r)^R \right) - \left( (V_r^t - a)\rho^L - (\rho V_r)^L \right)}{2a}
\]

\[
\rho* = \frac{\left( (V_a^t + a)\rho^R - (\rho V_a)^R \right) - \left( (V_a^t - a)\rho^L - (\rho V_a)^L \right)}{2a}
\]

and:

\[
\frac{dp}{\tau_r} = \frac{dpV_a}{1}; \quad \text{across } \frac{dx}{dt} = V_a - a \\
\frac{dp}{\tau_r} = \frac{dpV_a}{1}; \quad \text{across } \frac{dx}{dt} = V_a + a
\]

where the * indicates variables at the cell interface and the superscripts L and R indicate the variables at the left and right of the cell interface, respectively (Fig. 4). The superscript t denotes explicit variables (i.e. variables evaluated at the previous time step).

B. Second-order of accuracy: MUSCL-Hancock approach

To achieve second-order accuracy in space, linear reconstruction within each cell is required (Fig. 5). Since directional splitting operator is used for the Riemann solution (Eqs (A1) and (A2)), the sub-cell reconstruction is also split along the radial and axial direction. Knowing the cell-averaged data at time level \( t \), the linear reconstruction is realized by considering the following linear cell-distribution:

\[
\mathbf{w}(r, x) = \mathbf{U}_{ij} + \frac{(r - r_i)}{\Delta r} \mathbf{\zeta_i}; \quad \mathbf{w}(r, x) = \mathbf{U}_{ij} + \frac{(x - x_j)}{\Delta x} \mathbf{\zeta_j}
\]

where \( \mathbf{w} \) are the linear reconstructed data. The \( \zeta_i \) and \( \zeta_j \) are the slopes along the r and x direction, respectively. A VanLeer slope limiter (Toro, 2009) is used to compute those slopes to eliminate spurious oscillations given by:

\[
\zeta_i(\Delta^+, \Delta^-) = [\text{sgn}(\Delta^+) + \text{sgn}(\Delta^-)] \frac{|\Delta^+|}{|\Delta^+| + |\Delta^-| + \xi_d} \tag{A12}
\]

where:

\[
\Delta^+ = \frac{U_{i+1,j} - U_{ij}}{\Delta r}; \quad \Delta^- = \frac{U_{ij} - U_{i-1,j}}{\Delta r}
\]

and:

\[
\left\{ \begin{array}{l}
\frac{dp}{\tau_r} = \frac{dpV_r}{1}; \quad \text{across } \frac{dx}{dt} = V_r - a \\
\frac{dp}{\tau_r} = \frac{dpV_r}{1}; \quad \text{across } \frac{dx}{dt} = V_r + a
\end{array} \right. \quad \text{along the radial direction} \tag{A8}
\]
\( \varepsilon_2 \) is a very small number to avoid singularity; and \( \text{sgn} \) is the sign function. \( \beta_j \) could be evaluated in a similar way. Notice that \( \mathcal{U} \) in Eq. (A13) could be any primitive or conservative variable. In this work, the pressure \( P \) is chosen to compute the jumps \( \Delta^R \) and \( \Delta^- \). Another slope limiter, MINMOD (Toro, 2009), is also used and showed almost similar results to the VanLeer slope limiter.

Notice from Figures 3 and 5 that what is important from the reconstruction is to obtain the values at the cell interface which will be considered as left/right states for the Riemann problem. Therefore, assuming a rectangular mesh and evaluating Eq. (A11) at the cell interfaces \( i + 1/2 \) and \( j + 1/2 \), respectively, gives the left and right states for the Riemann problem as follows:

along the radial direction :
\[
\begin{align*}
\varpi^L_{i+1/2,j} &= \mathcal{U}^l_{i,j} + \frac{1}{2} \beta_i; \\
\varpi^R_{i+1/2,j} &= \mathcal{U}^r_{i+1,j} - \frac{1}{2} \beta_{i+1}
\end{align*}
\tag{A14}
\]

along the axial direction :
\[
\begin{align*}
\varpi^L_{i,j+1/2} &= \mathcal{U}^l_{i,j} + \frac{1}{2} \beta_j; \\
\varpi^R_{i,j+1/2} &= \mathcal{U}^r_{i+1,j} - \frac{1}{2} \beta_{j+1}
\end{align*}
\tag{A15}
\]

where the superscripts \( L \) and \( R \) indicate left and right states for a Riemann problem at a cell interface (i.e. at \( r_{i+1/2} \) and \( x_{j+1/2} \)), respectively.

To achieve second-order accuracy in space and time, the Hancock approach (Toro, 2009) is used, where the left and right states in Eqs (A14) and (A15) are evolved in time by half time step (at \( \Delta t/2 \)), then the Riemann problem is considered using the new updated states. The procedure of Hancock approach is summarized in the following steps:

i. From the linear reconstruction (Eq. (A11)), the left and right states for a Riemann problem are obtained (Eqs (A14) and (A15)).

ii. Evolution of the left and right states (Eqs (A14) and (A15)) by half time step as follows:

along the radial direction :
\[
\begin{align*}
\varpi^L_{i+1,j} &= \mathcal{U}^l_{i,j} + \frac{1}{2} \beta_i + \frac{1}{2} \Delta \beta^R \left[ \mathcal{F} \left( \frac{\varpi^R_{i+1/2,j}}{2} \right) - \mathcal{F} \left( \frac{\varpi^L_{i+1/2,j}}{2} \right) \right] \\
\varpi^R_{i+1,j} &= \mathcal{U}^r_{i,j} + \frac{1}{2} \beta_i + \frac{1}{2} \Delta \beta^L \left[ \mathcal{F} \left( \frac{\varpi^R_{i+1/2,j}}{2} \right) - \mathcal{F} \left( \frac{\varpi^L_{i+1/2,j}}{2} \right) \right]
\end{align*}
\tag{A16}
\]

iii. The Riemann problem is considered using the new states in Eqs (A16) and (A17) where the Riemann solution (Eqs (8) and (9)) is used to compute the fluxes in Eq. (6) at the cell interfaces.

C. Third order of accuracy: WENO reconstruction

Similarly to the previous linear reconstruction, directional splitting will be used. In this case, to achieve third order of accuracy, one dimensional quadrature reconstruction is required within each control volume as follows:

along the \( x \) - direction :
\[
\begin{align*}
\varpi \left( r_i,x \right) &= U_{i,j} + \frac{\beta_i}{2} \left( x - x_j \right) + \frac{\beta_i}{2} \left( x - x_j \right)^2
\end{align*}
\tag{A18}
\]

For consistency and conservation within a finite volume (\( \Omega_{i,j} \)), the following three conditions must be fulfilled:

\[
\begin{align*}
\mathcal{U}_{i,j} &= \frac{1}{\Delta x} \int_{x_{j-1/2}}^{r_{i+1/2}} \varpi \left( r,x \right) \, dx; \\
U \left( r_i,x_{j+1/2} \right) &= U_{i,j} + \frac{\Delta \varpi^L_{i,j}}{2}; \\
U \left( r_i,x_{j-1/2} \right) &= U_{i,j} + \frac{\Delta \varpi^R_{i,j}}{2}
\end{align*}
\tag{A19}
\]

which result in:

\[
\begin{align*}
U_{i,j} &= \frac{3}{2} \mathcal{U}_{i,j} - \frac{1}{4} \left( \Delta \varpi^R_{i,j} + \Delta \varpi^L_{i,j} \right); \\
\beta_i &= \frac{\Delta \varpi^R_{i,j} + \Delta \varpi^L_{i,j}}{\Delta x}; \\
\beta_j &= \frac{6 \left( \Delta \varpi^R_{i,j} + \Delta \varpi^L_{i,j} - 2 \mathcal{U}_{i,j} \right)}{\Delta x^2}
\end{align*}
\tag{A20}
\]

Similar solutions could be obtained for the case along the \( r \)-direction. It is important to notice that in Eq. (A19) the indexes \( L \) and \( R \) are used to indicate the left and right states of either the control volume or the cell interface based on whether the variable \( U \) is given at the cell centre (e.g. \( U_{i,j} \)) or at the a cell interface (e.g. \( U_{i,j+1/2} \)), respectively.

As stated in the previous section, what is important from a reconstruction is to obtain the values at both sides of the cell interfaces (Figs 3 and 5). Basically, the WENO reconstruction (Jiang & Shu, 1996) is a total variation diminishing (TVD) technique which determines those left and right states while avoiding spurious oscillation and maintaining the desired order of accuracy of the scheme. Considering the axial direction, the
left and right states of the cell centred at \(j\) from a WENO reconstruction are given as follows (Jiang & Shu, 1996):

\[
\omega_j^R = \sum_{s=0}^{2} N_j^R \lambda_s^R; \quad \omega_j^L = \sum_{s=0}^{2} N_j^L \lambda_s^L \quad (A21)
\]

where:

\[
\lambda_0^R = \frac{1}{3} U_j + \frac{5}{6} U_{j+1} - \frac{1}{6} U_{j+2};
\lambda_1^R = -\frac{1}{6} U_{j-1} + \frac{5}{6} U_j + \frac{1}{3} U_{j+1};
\lambda_2^R = \frac{1}{3} U_{j-2} - \frac{7}{6} U_{j-1} + \frac{11}{6} U_j
\]

\[
\lambda_0^L = \frac{11}{6} U_j - \frac{7}{6} U_{j+1} + \frac{1}{3} U_{j+2};
\lambda_1^L = \frac{1}{3} U_{j-1} + \frac{5}{6} U_j - \frac{1}{6} U_{j-2};
\lambda_2^L = -\frac{1}{6} U_{j-2} + \frac{5}{6} U_{j-1} + \frac{1}{3} U_j \quad (A22)
\]

\[
N_j^R = \frac{h_j^R}{\sum_{m=0}^{2} h_m^R}; b_j^R = \frac{a_j^R}{(\beta_j + \varepsilon)};
\]

\[
N_j^L = \frac{h_j^L}{\sum_{m=0}^{2} h_m^L}; b_j^L = \frac{a_j^L}{(\beta_j + \varepsilon)} \quad (A24)
\]

\[
\begin{align*}
\beta_0 &= \frac{13}{12} (U_j - 2U_{j+1} + U_{j+2}) + \frac{1}{4} (3U_j - 4U_{j+1} + U_{j+2})^2 \\
\beta_1 &= \frac{13}{12} (U_{j-1} - 2U_j + U_{j+1}) + \frac{1}{4} (3U_{j-1} - 4U_j + U_{j+1})^2 \\
\beta_2 &= \frac{13}{12} (U_{j-2} - 2U_{j-1} + U_j) + \frac{1}{4} (3U_{j-2} - 4U_{j-1} + U_j)^2 \\
\end{align*} \quad (A25)
\]

And

\[
\tilde{\alpha}_0 = \frac{3}{10} \quad ; \quad \tilde{\alpha}_1 = \frac{3}{8} \quad ; \quad \tilde{\alpha}_2 = \frac{1}{10} \quad (A26)
\]

In Eq. (A24), \(\varepsilon\) is a “calibration” parameter to avoid singularities. A similar procedure (Eqs (A21) – (A26)) is applied for the radial direction.

As seen from the above procedure, the WENO method uses smoothness parameters (\(\tilde{\alpha}\) and \(\lambda\)) over five consecutive cells (discrete control volumes) which, under smooth transient flow, will achieve fifth order of accuracy (Jiang & Shu, 1996; Luo et al., 2013). However, in non-smooth transient flow, the scheme achieves only third-order accuracy. Such a scheme is usually referred to as third/fifth-order scheme. However, this work refers to this scheme as fifth-order scheme (its maximum order of accuracy) and its accuracy is tested later.

### D. Second-order WENO scheme

The above WENO procedure (Eqs (A21) – (A26)) could be, similarly, applied to obtain a second-order WENO scheme (Jiang & Shu, 1996) with linear reconstruction. In this case, the WENO method uses smoothness parameters over three consecutive cells (finite volumes) which, under smooth transient flow, achieves third order of accuracy. However, in non-smooth transient flow, the scheme achieves only second order of accuracy. Such a scheme is usually referred to as a second/third-order scheme. However, this work refers to this scheme as a third-order scheme (its maximum order of accuracy) and its accuracy is tested later.

### E. Characteristic equation for CBC

The CBC method is implemented at the upstream \((x = 0)\) and downstream boundaries \((x = L)\) of the pipe, and therefore only the axial direction is considered. The split system of equations along the axial direction (Eq. (A2)) is rewritten using the characteristic variables \((\bar{\Psi}_x)\) as follows (Toro, 2009):

\[
\frac{\partial \bar{\Psi}_x^{ID}}{\partial t} + \Lambda^G \frac{\partial \bar{\Psi}_x^{ID}}{\partial x} = 0 \quad (A27)
\]

where:

\[
\bar{\Psi}_x^{ID} = L_G \bar{U}_x^{ID}; \quad \Lambda^G = L_G J_G L_G^{-1} = \begin{bmatrix} L \gamma^G & 0 \\ 0 & L \gamma^G \end{bmatrix} \quad (A28)
\]

and \(L_G\) is the left eigenvector of \(J_G\) which is:

\[
L_G = \begin{bmatrix} -V_x - a & 1 \\ V_x - a & 1 \end{bmatrix} \quad (A29)
\]

and its inverse is:

\[
L_G^{-1} = \begin{bmatrix} -\frac{1}{V_x-a} & \frac{1}{V_x-a} \\ -\frac{V_x-a}{a} & \frac{V_x-a}{a} \end{bmatrix} \quad (A30)
\]

The characteristic lines are given by (Toro, 2009):

\[
\bar{\Psi}_x^{ID} = L_G \bar{U}_x^{ID} = 0
\]

\[
\Rightarrow \begin{cases} 
 d(\rho V_x) - (V_x + a)d\rho = 0; & \text{if } \frac{dx}{dt} = V_x - a \\
 d(\rho V_x) - (V_x - a)d\rho = 0; & \text{if } \frac{dx}{dt} = V_x + a
\end{cases} \quad (A31)
\]

which after simplification gives:

\[
\begin{cases} 
 d\frac{V_x}{a} - \frac{d\rho}{\rho} = 0; & \text{if } \frac{dx}{dt} = V_x - a \\
 d\frac{V_x}{a} + \frac{d\rho}{\rho} = 0; & \text{if } \frac{dx}{dt} = V_x + a
\end{cases} \quad (A32)
\]
Integrating Eq. (A32), yields:

\[
\begin{cases}
V_x - a \log(\rho) = C^-; \quad \text{if} \quad \frac{dx}{dt} = V_x - a \\
V_x + a \log(\rho) = C^+; \quad \text{if} \quad \frac{dx}{dt} = V_x + a
\end{cases}
\] (A33)

where \(C^-\) and \(C^+\) are two constants called the Riemann invariants.

**Notation**

- \(a\) = wave speed (m s\(^{-1}\))
- \(b\) = scheme order of accuracy (–)
- \(b_0\) = a reference scheme order of accuracy (–)
- \(D\) = pipe diameter (m)
- \(E_F\) = energy flux (J)
- \(E_n\) = reflected energy from the boundary for a given mode \(n\) (J)
- \(E_n\) = energy content in a given mode \(n\) (J)
- \(E_T\) = total injected wave energy (J)
- \(F_i\) = vector-fluxes at cell interfaces (Pa)
- \(f\) = frequency (Hz)
- \(f_c\) = central frequency (Hz)
- \(f_n\) = cut-off frequencies of radial modes (Hz)
- \(H_0\) = initial pressure head (m)
- \(K\) = bulk modulus of water (Pa)
- \(L\) = pipe length (m)
- \(M\) = Mach number (–)
- \(N^b_S\) = averaged \(N_S\) at a given order of accuracy \(b\) (–)
- \(n\) = mode number (–)
- \(n_a\) = number of fictitious cells added at the boundaries (–)
- \(N_r\), \(N_z\) = number of cells along the radial and axial directions, respectively (–)
- \(N_S\) = number of discretized cells per central wavelength (–)
- \(n_a\) = indicator of inviscid or viscous case (–)
- \(P\) = pressure (Pa)
- \(P_0\) = initial pressure in the pipe (Pa)
- \(P_F\) = pressure at the source (Pa)
- \(P_{Jou}\) = Joukowski pressure (Pa)
- \(P_{res}\) = fixed pressure at the reservoir (Pa)
- \(P_s\) = maximum pressure induced by the wave at the source (Pa)
- \(R\) = pipe radius (m)
- \(R_e\) = Reynolds number (–)
- \(r\) = radial coordinate (m)
- \(S\) = vector of source term
- \(\tau^b_{CPU}\) = fitted computational time function for a given scheme of order \(b\) (s)
- \(t\) = time (s\(^{-1}\))
- \(t_{closure}\) = time closure (s)

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