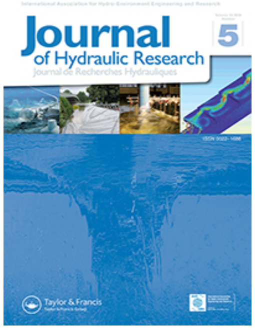

Journal of Hydraulic Research

ISSN: 0022-1686 (Print) 1814-2079 (Online) Journal homepage: https://www.tandfonline.com/loi/tjhr20

Does the stream power theory have a physical foundation?

William G. Gray, Mohamed S. Ghidaoui & Bryan W. Karney

Journal of Hydraulic Research Vol. 56, No. 5 (2018), pp. 585–595
https://doi.org/10.1080/00221686.2017.1405368

Forum paper

Does the stream power theory have a physical foundation?

WILLIAM G. GRAY, Professor, *Environmental Sciences and Engineering, University of North Carolina, Chapel Hill, NC 27599-3431, USA*
Email: GrayWG@unc.edu (author for correspondence)

MOHAMED S. GHIDAOUI, Chair Professor, Chinese Estates Professor of Engineering, *Department of Civil and Environmental Engineering, The Hong Kong University of Science and Technology, Hong Kong*

BRYAN W. KARNEY, Professor, *Department of Civil Engineering, University of Toronto, Toronto, Canada*

ABSTRACT
The minimum entropy production principle (mEPP) for flow in a wide rectangular channel shows that the entropy production rate is equal to the unit stream power divided by the product of temperature and flow depth. Only for isothermal, steady, uniform flow is an extremum of the entropy production equivalent to an extremum of unit stream power. For a sediment-carrying channel, minimization of unit stream power provides a linear relation between channel slope and water depth. In agreement with published results, the unit stream power is shown to be constant rather than minimized by mEPP within accuracy provided by data. The entropy generation rate is minimized only when the flow is as deep as possible. For a specified volumetric flow rate, the entropy production rate for uniform flow is is less than that for an M2, M3, or S3 profile but greater than that for an M1, S1, or S2 profile.

Keywords: Channel thermodynamics; minimum entropy production; morphodynamics and channel forms; river channels; stream power

1 Introduction

1.1 *Statement of the problem and aim of this paper*

Yang (1976, 1994) hypothesize that when a sedimentary channel is subjected to a fixed flow, the slope of the channel will adjust so that the entropy production rate will be minimized at a stationary state of normal flow for the channel. Parker (1977) compared the theory proposed in Yang (1976) with other theories. He concluded that the minimization theory produces solutions that seem reasonable for small slopes, but it fails for larger channel slopes. Yang (1996) reported that his theory provides acceptable results in the low flow region, but its accuracy diminishes as the Froude number increases and is unacceptable for supercritical flow. That the minimization theory appears to be acceptable when Froude number is small will be shown to be due to the fact that the entropy generation rate is minimized when the depth of flow is large and velocity small. Furthermore, our analysis of the equations and data for the Rio Grande River suggests that minimizing power is ad hoc at best and that the results obtained and subsequent conclusions are only coincidentally correct in special cases.

In the opening of his discussion of Yang (1976), Parker (1977) writes

> Many researchers, including the writer, remain wary of the unit stream power minimization principle utilized by the author insofar as it appears to have been "pulled out of the hat," with no physical justification.

The aim of the current paper is to investigate whether the minimum stream power hypothesis as it has been employed has a physical foundation. We stress that we are not advocating use of a different principle; our only objective is to determine the range of applicability of the currently employed application of a minimum energy production principle (mEPP) to stream flow hydraulics. Certainly, there have been numerous publications in the last 40 years advocating use of the minimum stream power theory. In addition, this theory constitutes a major theme of a book on sediment transport (Yang, 1994). There are indications that this theory is being progressively accepted among engineers.

Yet, the question as to whether this theory is, as some have claimed, "pulled out of the hat" remains open. Our paper addresses only this issue. That is, we ask whether we can derive

Received 17 June 2016; accepted 12 November 2017/Open for discussion until 31 March 2019.
This article has been republished with minor changes. These changes do not impact the academic content of the article.

ISSN 0022-1686 print/ISSN 1814-2079 online
http://www.tandfonline.com

the minimum stream power theory. We approach this objectively, from first principles. We apply the fundamental laws of mass, momentum, energy and entropy conservation to one-dimensional channel flow with rigour and see where this leads us in relation to the minimum stream power theory. There are a number of entropy-based theories, such as information entropy and Tsallis entropy, being proposed for water resources applications; but such theories are beyond the scope of the current paper (although we do plan to look into other entropy theories in the future).

2 Expression for entropy generation for open channel flow

Gray and Ghidaoui (2009) provide a rigorous derivation of the entropy generation expression for open channel flow based on the approaches of continuum mechanics (Eringen, 1980; Truesdell, 1984), classical irreversible thermodynamics (De Groot & Mazur, 1984; Jou & Casas-Vázquez, 2001), and thermodynamically constrained averaging theory (TCAT) (Gray & Miller, 2013). However, in maintaining the rigour, some of the insights are likely obfuscated for many readers. Therefore, the current analysis uses simpler, more-familiar equations that nevertheless retain all elements that allow for the analysis to be valid. We consider the equations for open channel flow in a wide rectangular channel when variation of the velocity field over the cross section can be neglected. We also neglect leakage over the wetted perimeter of the channel and inflow and evaporation over the upper surface. Curvature of the channel is also neglected. We emphasize that these simplifications are not intrinsic to the analysis but help clarify the heart of the analysis. Furthermore, we have made use of average quantities defined systematically so that some averages are weighted to ensure that the equations are consistent. For example, the average temperature is an entropy weighted average. The precise definitions of averages are not essential to understanding the following development and are not indicated, but the interested reader may find the definitions explicitly listed in Gray and Ghidaoui (2009).

2.1 *Mass conservation*

The three-dimensional point mass conservation equation is well known:

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho \boldsymbol{v}) = 0 \tag{1}$$

where ρ is the fluid density and $\boldsymbol{v}$ is the velocity vector. This equation may be integrated over the cross section of the channel of constant width, B, much greater than the depth, H, using integral theorems (Gray, Leijnse, Kolar, & Blain 1993), as well as over an increment of time if desired, to obtain:

$$\frac{\partial(\overline{\rho}H)}{\partial t} + \frac{\partial(\overline{\rho}UH)}{\partial x} = 0 \tag{2}$$

where $\overline{\rho}$ is the average fluid density and U is the density weighted average velocity in the direction of flow, x. The overbar is used to indicate that the quantity has been averaged over the channel cross-section, and an increment of time. Capitalization rather than an overbar is used to indicate the average of the velocity component in the direction of flow because this is a density weighted average. Equation (2) may be rearranged so that the time variation is a Lagrangian derivative along the channel axis with the form:

$$\frac{\mathrm{D}(\overline{\rho}H)}{\mathrm{D}t} + \overline{\rho}H\frac{\partial U}{\partial x} = 0 \tag{3}$$

where:

$$\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + U\frac{\partial}{\partial x} \tag{4}$$

2.2 *Momentum conservation*

The momentum equation at a point is:

$$\frac{\partial(\rho \boldsymbol{v})}{\partial t} + \nabla\cdot(\rho\boldsymbol{v}\boldsymbol{v}) - \rho\boldsymbol{g} - \nabla\cdot\mathbf{t} = 0 \tag{5}$$

where $\mathbf{g}$ is gravity and $\mathbf{t}$ is the total stress tensor. Equation (5) is a vector equation, but we have primary interest in the momentum transport along the channel for the case when the hydrostatic condition applies for the momentum in the vertical. Integration of this equation over the cross section of the wide rectangular channel yields

$$\frac{\partial(\overline{\rho}UH)}{\partial t} + \frac{\partial(\overline{\rho}UUH)}{\partial x} - \overline{\rho}gH\left(S_0 - S_f\right) - \frac{\partial(\bar{\bar{t}}_{xx}H)}{\partial x} = 0 \tag{6}$$

where S_0 is the channel slope, S_f is the friction slope (the retarding force due to interaction of flow with the channel boundary), and $\bar{\bar{t}}_{xx}$ is the stress for the flow channel which also accounts for the stress arising from the inertial term due to averaging of velocity squared such that:

$$\bar{\bar{t}}_{xx} = \overline{t_{xx} - \rho(u-U)(u-U)} \tag{7}$$

where t_{xx} is the point stress in the flow along the channel and the additional terms on the right account for turbulent stress as well as stress due to non-uniformity of the velocity across the channel cross section. Rearrangement of the time derivative in Eq. (6) into a Lagrangian derivative form gives the momentum conservation equation:

$$\frac{\mathrm{D}(\overline{\rho}UH)}{\mathrm{D}t} + \overline{\rho}UH\frac{\partial U}{\partial x} - \overline{\rho}gH\left(S_0 - S_f\right) - \frac{\partial(\bar{\bar{t}}_{xx}H)}{\partial x} = 0 \tag{8}$$

2.3 *Energy conservation*

The point form of the total energy equation is perhaps less commonly used in hydraulics than the preceding conservation equations, but it is still foundational. It can be expressed as:

$$\frac{\partial}{\partial t}\left(E+\rho\frac{\boldsymbol{v}\cdot\boldsymbol{v}}{2}\right)+\nabla\cdot\left[\left(E+\rho\frac{\boldsymbol{v}\cdot\boldsymbol{v}}{2}\right)\boldsymbol{v}\right] - \rho\boldsymbol{g}\cdot\boldsymbol{v}-h-\nabla\cdot(\mathbf{t}\cdot\boldsymbol{v})-\nabla\cdot\boldsymbol{q}=0 \tag{9}$$

where E is the internal energy per volume, $\mathbf{q}$ is the non-advective heat transfer vector, and h is an energy body source. Integration of this equation while using the assumption that the impact of the velocity components orthogonal to the channel axis are negligible yields:

$$\frac{\partial}{\partial t}\left[\left(\overline{E}+\overline{\rho}\frac{UU}{2}\right)H\right]+\frac{\partial}{\partial x}\left[\left(\overline{E}+\overline{\rho}\frac{UU}{2}\right)UH\right] - \overline{\rho}gUS_0H-\overline{h}H-\frac{\partial(U\overline{\overline{t}}_{xx}H)}{\partial x}-\frac{\partial(\overline{\overline{q}}_xH)}{\partial x}=0 \tag{10}$$

In this equation the deviation kinetic energy is omitted, as is customary in the inertial terms; the terms that arise for energy transport due to deviation velocities associated with turbulence and variability across the channel cross section are included in $\overline{\overline{q}}_x$, the non-advective energy transport along the channel; and non-advective energy transport across the wetted perimeter and the top of the channel are considered negligible. Rewriting Eq. (10) in terms of the Lagrangian derivative provides:

$$\frac{\mathrm{D}}{\mathrm{D}t}\left[\left(\overline{E}+\overline{\rho}\frac{UU}{2}\right)H\right]+\left[\left(\overline{E}+\overline{\rho}\frac{UU}{2}\right)H\right]\frac{\partial U}{\partial x} - \overline{\rho}gUS_0-\overline{h}H-\frac{\partial(U\overline{\overline{t}}_{xx}H)}{\partial x}-\frac{\partial(\overline{\overline{q}}_xH)}{\partial x}=0 \tag{11}$$

2.4 *Entropy equation*

In addition to the preceding three conservation equations, system behaviour is subject to the second law of thermodynamics as expressed by the entropy balance equation. Although this equation is somewhat rarely used in the context of hydraulic analysis, the point form of this balance is:

$$\frac{\partial\eta}{\partial t}+\nabla\cdot(\eta\boldsymbol{v})-\nabla\cdot\boldsymbol{\varphi}-b=\Lambda \tag{12}$$

where η is the entropy per volume, $\boldsymbol{\varphi}$ is the non-advective entropy flux, b is the body source of entropy, and Λ is the non-negative entropy production rate per volume. Averaging this equation over the channel cross-section, as well as a time increment if desired, gives:

$$\frac{\partial(\overline{\eta}H)}{\partial t}+\frac{\partial(\overline{\eta}UH)}{\partial x}-\frac{\partial(\overline{\overline{\varphi}}_xH)}{\partial x}-\overline{b}H=\overline{\Lambda}H \tag{13}$$

Once again, the double overbar used with φ_x indicates that this quantity is an average of the microscale non-advective entropy flux along the channel axis that includes contributions from velocity deviations. Additionally, the non-advective entropy transport across the boundary of the channel is considered negligible, consistent with the assumption in the energy equation that non-advective energy transport at the boundary is inconsequential. Written in terms of the Lagrangian time derivative, Eq. (13) takes the form:

$$\frac{\mathrm{D}(\overline{\eta}H)}{\mathrm{D}t}+\overline{\eta}H\frac{\partial U}{\partial x}-\frac{\partial(\overline{\overline{\varphi}}_xH)}{\partial x}-\overline{b}H=\overline{\Lambda} \tag{14}$$

2.5 *Classical irreversible thermodynamics*

The entropy equation plays a key physical role by placing a constraint on system behaviour; but to apply this constraint, a relation between entropy per volume and energy per volume must be established. This is obtained using a thermodynamic expression. Here, we make use of the the simplest form based on classical irreversible thermodynamics (CIT). This formulation is particularly appropriate because it satisfies the condition of local equilibrium required for mEPP as noted by Kay (2003) and Trepczyńska-Lent (2012). For CIT, the energy per volume is given by the Euler form (Bailyn, 1994; Callen, 1985):

$$E-\eta\theta+p-\rho\mu=0 \tag{15}$$

where θ is temperature, p is pressure, and μ is the chemical potential. The differential of the Euler equation is:

$$\mathrm{d}E-\theta\,\mathrm{d}\eta-\mu\,\mathrm{d}\rho=0 \tag{16}$$

The differential in this equation can be written both as a partial time derivative and as a gradient because of the local equilibrium assumption to obtain:

$$\frac{\partial E}{\partial t}-\theta\frac{\partial\eta}{\partial t}-\mu\frac{\partial\rho}{\partial t}=0 \tag{17}$$

and

$$\nabla E-\theta\nabla\eta-\mu\nabla\rho=0 \tag{18}$$

2.6 *Thermodynamic equation*

Gray (2002) and Gray and Miller (2009,1,1) have shown the importance of averaging thermodynamic relations from the smaller scale to the larger scale rather than merely posing the thermodynamics at the larger scale. If Eqs (17) and (18) are averaged and combined we obtain:

$$\frac{\mathrm{D}(\overline{E}H)}{\mathrm{D}t}-\overline{\theta}\frac{\mathrm{D}(\overline{\eta}H)}{\mathrm{D}t}-\overline{\mu}\frac{\mathrm{D}(\overline{\rho}H)}{\mathrm{D}t} + \eta\overline{\frac{\mathrm{D}(\theta-\overline{\theta})}{\mathrm{D}t}H}+\rho\overline{\frac{\mathrm{D}(\mu-\overline{\mu})}{\mathrm{D}t}H} \tag{19}$$

In this equation, $\overline{\mu}$ is the density weighted average of the chemical potential and $\overline{\theta}$ is the entropy weighted average temperature.

In both cases, the averaging is over the channel cross section and an increment in time, if desired.

The preceding conservation and thermodynamic equations are employed, making use of Lagrange multipliers, as constraints on the entropy balance equation. The result is the rather formidable combined expression:

$$
\begin{aligned}
&\frac{\mathrm{D}(\overline{\eta}H)}{\mathrm{D}t}+\overline{\eta}H\frac{\partial U}{\partial x}-\frac{\partial(\overline{\overline{\varphi}}_xH)}{\partial x}-\overline{b}H\\
&+\lambda_T\left\{\frac{\mathrm{D}(\overline{E}H)}{\mathrm{D}t}-\overline{\theta}\frac{\mathrm{D}(\overline{\eta}H)}{\mathrm{D}t}-\overline{\mu}\frac{\mathrm{D}(\overline{\rho}H)}{\mathrm{D}t}\right.\\
&\qquad\left.+\,\overline{\eta\frac{\mathrm{D}(\theta-\overline{\theta})}{\mathrm{D}t}}H+\overline{\rho\frac{\mathrm{D}(\mu-\overline{\mu})}{\mathrm{D}t}}H\right\}\\
&+\lambda_E\left[\frac{\mathrm{D}}{\mathrm{D}t}\left\{\left[\overline{E}+\overline{\rho}\frac{UU}{2}\right]H\right\}+\left\{\left[\overline{E}+\overline{\rho}\frac{UU}{2}\right]H\right\}\frac{\partial U}{\partial x}\right]\\
&-\lambda_E\left[\overline{\rho}gUS_0H+\overline{h}H+\frac{\partial(U\overline{\overline{t}}_{xx}H)}{\partial x}+\frac{\partial(\overline{\overline{q}}_xH)}{\partial x}\right]\\
&+\lambda_P\left[\frac{\mathrm{D}(\overline{\rho}UH)}{\mathrm{D}t}+\overline{\rho}UH\frac{\partial U}{\partial x}-\overline{\rho}gH\left(S_0-S_f\right)-\frac{\partial(\overline{\overline{t}}_{xx}H)}{\partial x}\right]\\
&+\lambda_M\left[\frac{\mathrm{D}(\overline{\rho}H)}{\mathrm{D}t}+\overline{\rho}H\frac{\partial U}{\partial x}\right]=\overline{\Lambda}H \qquad (20)
\end{aligned}
$$

To progress, we seek to eliminate the material derivatives by appropriate selection of the Lagrange multipliers indicated as λs in the preceding equation. This will leave the entropy production rate on the right side of the equation expressed in terms of the dissipative, irreversible fluxes and the forces that drive these fluxes. Therefore choose:

$$
\{\lambda_T,\lambda_E,\lambda_P,\lambda_M\}=\left\{\frac{1}{\overline{\overline{\theta}}},-\frac{1}{\overline{\overline{\theta}}},\frac{U}{\overline{\overline{\theta}}},\frac{1}{\overline{\overline{\theta}}}\left(\overline{\mu}-\frac{U^2}{2}\right)\right\} \qquad (21)
$$

Substituting these forms into the constrained entropy Eq. (20) results in a new, but still rather intimidating, form:

$$
\begin{aligned}
&\frac{\mathrm{D}(\overline{\eta}H)}{\mathrm{D}t}+\overline{\eta}H\frac{\partial U}{\partial x}-\frac{\partial(\overline{\overline{\varphi}}_xH)}{\partial x}-\overline{b}H\\
&+\frac{1}{\overline{\overline{\theta}}}\left\{\frac{\mathrm{D}(\overline{E}H)}{\mathrm{D}t}-\overline{\theta}\frac{\mathrm{D}(\overline{\eta}H)}{\mathrm{D}t}-\overline{\mu}\frac{\mathrm{D}(\overline{\rho}H)}{\mathrm{D}t}\right.\\
&\qquad\left.+\,\overline{\eta\frac{\mathrm{D}(\theta-\overline{\theta})}{\mathrm{D}t}}H+\overline{\rho\frac{\mathrm{D}(\mu-\overline{\mu})}{\mathrm{D}t}}H\right\}\\
&-\frac{1}{\overline{\overline{\theta}}}\left[\frac{\mathrm{D}}{\mathrm{D}t}\left\{\left[\overline{E}+\overline{\rho}\frac{UU}{2}\right]H\right\}+\left\{\left[\overline{E}+\overline{\rho}\frac{UU}{2}\right]H\right\}\frac{\partial U}{\partial x}\right]\\
&+\frac{1}{\overline{\overline{\theta}}}\left[\overline{\rho}gUS_0H+\overline{h}H+\frac{\partial(U\overline{\overline{t}}_{xx}H)}{\partial x}+\frac{\partial(\overline{\overline{q}}_xH)}{\partial x}\right]\\
&+\frac{U}{\overline{\overline{\theta}}}\left[\frac{\mathrm{D}(\overline{\rho}UH)}{\mathrm{D}t}+\overline{\rho}UH\frac{\partial U}{\partial x}-\overline{\rho}gH\left(S_0-S_f\right)-\frac{\partial(\overline{\overline{t}}_{xx}H)}{\partial x}\right]\\
&+\frac{1}{\overline{\overline{\theta}}}\left[\overline{\mu}-\frac{U^2}{2}\right]\left[\frac{\mathrm{D}(\overline{\rho}H)}{\mathrm{D}t}+\overline{\rho}H\frac{\partial U}{\partial x}\right]=\overline{\Lambda}H \qquad (22)
\end{aligned}
$$

However, cancelling and combing terms allows a little more clarity to emerge:

$$
\begin{aligned}
&-\frac{\partial}{\partial x}\left(\overline{\overline{\varphi}}_xH-\frac{\overline{\overline{q}}_xH}{\overline{\theta}}\right)+\frac{\overline{\overline{q}}_xH}{\overline{\theta}^2}\frac{\partial\overline{\theta}}{\partial x}\\
&-\left\{\overline{b}-\frac{1}{\overline{\overline{\theta}}}\left[\overline{h}+\overline{\eta\frac{\mathrm{D}(\theta-\overline{\theta})}{\mathrm{D}t}}+\overline{\rho\frac{\mathrm{D}(\mu-\overline{\mu})}{\mathrm{D}t}}\right]\right\}H\\
&-\frac{1}{\overline{\overline{\theta}}}\left[\left\{\left[\overline{E}-\overline{\eta}\overline{\theta}-\overline{\rho}\overline{\mu}-t_{xx}\right]H\right\}\frac{\partial U}{\partial x}\right]\\
&+\frac{U}{\overline{\overline{\theta}}}\left(\overline{\rho}gHS_f\right)=\overline{\Lambda}H \qquad (23)
\end{aligned}
$$

The next step uses the concept of a simple system, as defined by Eringen (1980), as having two specific properties. The first is that the non-advective heat flux divided by the temperature is equal to the non-advective entropy flux such that:

$$
\overline{\overline{\varphi}}_xH-\frac{\overline{\overline{q}}_xH}{\theta}=0 \qquad (24)
$$

The second property is that the energy body source term divided by the temperature is equal to the entropy body source term. For the case where we have integrated to a larger scale, this relation also includes the fluctuations in temperature and chemical potential that dissipate as the system moves to an equilibrium configuration. This equality is stated as:

$$
\overline{b}-\frac{1}{\overline{\overline{\theta}}}\left[\overline{h}+\overline{\eta\frac{\mathrm{D}(\theta-\overline{\theta})}{\mathrm{D}t}}+\overline{\rho\frac{\mathrm{D}(\mu-\overline{\mu})}{\mathrm{D}t}}\right]=0 \qquad (25)
$$

We also invoke the averaged form of the local Euler relation at a point given by Eq. (15) which is:

$$
\overline{E}-\overline{\eta}\overline{\theta}-\overline{\rho}\,\overline{\mu}=-\overline{p} \qquad (26)
$$

Use of these relations in Eq. (23) and division by H simplifies it further to:

$$
\frac{\overline{\overline{q}}_x}{\overline{\overline{\theta}}^2}\frac{\partial\overline{\theta}}{\partial x}+\frac{1}{\overline{\overline{\theta}}}\left(\overline{p}+\overline{\overline{t}}_{xx}\right)\frac{\partial U}{\partial x}+\frac{1}{\overline{\overline{\theta}}}\left(\overline{\rho}gS_f\right)U=\overline{\Lambda} \qquad (27)
$$

This equation expresses the product of dissipative fluxes with their conjugate driving forces. Each force and flux term will be zero at equilibrium. The first product on the left involves the non-advective heat flux, $\overline{\overline{q}}_x$, multiplying the temperature gradient, $\partial\overline{\theta}/\partial x$. The second product on the left is the non-equilibrium fluid stress tensor, $\overline{\overline{t}}_{xx}+\overline{p}$, multiplying the velocity gradient, $\partial U/\partial x$. The third term is the friction slope, S_f, which multiplies the velocity U. The sum of these three products, multiplied by $1/\overline{\theta}$, is equal to the entropy generation rate per volume. If the system is isothermal and we ignore viscous effects within the fluid, the first two products of terms in this equation are zero. Thus we have the final form of the entropy inequality:

$$\frac{U}{\overline{\theta}}\left(\overline{\rho} g S_f\right) = \overline{\Lambda} \tag{28}$$

If the volumetric flow rate is denoted as $Q = UHW$ where W is the width of the channel, this equation can be rearranged to:

$$\frac{\overline{\rho} g Q}{\overline{\theta} W}\frac{S_f}{H} = \overline{\Lambda} \tag{29}$$

This result was also obtained by Gray and Ghidaoui (2009) using a more detailed analysis. It is this expression for entropy generation that must be examined in light of the mEPP. Alternatively, if the stream power, P, is defined as in Yang (1994) as:

$$P = \overline{\rho} g U S_f = \frac{\overline{\rho} g Q}{W}\frac{S_f}{H} \tag{30}$$

We can write Eq. (29) as:

$$\frac{1}{\overline{\theta}} P = \overline{\Lambda} \tag{31}$$

such that the stream power is seen to be directly proportional to the entropy generation rate. If the volumetric flow rate, Q, the channel width, W, and the average density $\overline{\rho}$ are constant, Eq. (29) significantly shows that S_f/H is proportional to the entropy production, $\overline{\Lambda}$, and therefore, of course, proportional to the stream power, P.

Note that Eq. (29) is valid for both steady and unsteady one-dimensional open channel flows. In deriving this equation, mass, momentum, and energy conservation were employed as constraints on the entropy inequality. Therefore, any additional condition on Eq. (29), such as requiring that it be extremum, is beyond what the conservation laws demand. The equation only provides the rate of entropy generation.

The preceding derivation remains valid for the case of a water–sediment mixture, where the state variables (H, Q and U) are taken to be those of the mixture. At a given section of the channel, the total mass flux is the sum of the water mass flux and the sediment mass flux. That is, $\overline{\rho} Q = \overline{\rho}^w Q^w + \overline{\rho}^s Q^s$, where superscripts w and s stand for water and sediment, respectively, and $\overline{\rho}$ is the density of the mixture. As a result, Eq. (29) becomes:

$$\frac{g}{\overline{\theta} W}\left(\overline{\rho}^w Q^w + \overline{\rho}^s Q^s\right)\frac{S_f}{H} = \overline{\Lambda} \tag{32}$$

and the stream power is:

$$P = \left(\overline{\rho}^w Q^w + \overline{\rho}^s Q^s\right) g \frac{S_f}{WH} \tag{33}$$

From Eq. (31) we see that the stream power divided by temperature is equal to the entropy production rate. Therefore, when the flow is isothermal, steady, and uniform, an extremum of entropy production would correspond to an extremum of unit stream power. Equation (29) describes the entropy production based on larger scale quantities. If one wishes to ensure that smaller scale contributions to entropy production are properly accounted for, this can be accomplished through selection of the form of S_f and by inclusion of the coefficients that account for a variable velocity profile as in Gray and Ghidaoui (2009).

3 Is the minimum stream power theory "pulled out of a hat"?

Yang (1994) writes that the minimum entropy principle indicates, "For a closed and dissipative system, available energy can only decrease with respective to time". As a consequence, he further states, "If the rate of energy dissipation due to sediment transport is relatively small and the velocity distribution is fairly uniform":

$$\begin{aligned} US_0 &= \text{a minimum subject to } S_0 = f(Q, H; \boldsymbol{a}) \text{ and} \\ Q &= \text{ constant} \end{aligned} \tag{34}$$

where f is an empirically proposed function (Yang, 1976) in which $\boldsymbol{a}$ is a vector of parameters such as the median diameter of sediments, viscosity, terminal velocity of sediments, sediment concentration, critical velocity, etc. The particular function, f, proposed in Yang (1976) is given and discussed later in this section. The above constrained optimization for a channel with width W such that $Q = UHW$ (i.e. the case considered in Yang, 1976, 1996) is:

$$\begin{aligned} \frac{QS_0}{WH} &= \text{a minimum subject to } S_0 = f(Q, H; \boldsymbol{a}) \text{ and} \\ Q &= \text{constant} \end{aligned} \tag{35}$$

Note that the minimization of $(QS_0)/(WH)$ is the same as the minimization of the entropy, $\overline{\Lambda}$, and the power, P, since W, ρ and θ are all constant. Taking the derivative of QS_0/WH to determine its minimum gives:

$$\frac{Q\frac{\mathrm{d}S_0}{\mathrm{d}H}H - QS_0}{H^2} = 0 \tag{36}$$

which leads to:

$$\frac{\mathrm{d}S_0}{S_0} = \frac{\mathrm{d}H}{H} \tag{37}$$

This is easily integrated to obtain:

$$S_0 = C_1 H \tag{38}$$

where C_1 is a constant of integration. That is, minimization of the stream power (entropy production rate) for a fixed volumetric flow rate leads to the linear relation of Eq. (38). It is interesting to study the sediment transport equation used in Yang (1976, 1996) to determine if it differs from a linear relation between S_0 and H. This transport equation in Yang (1976, 1996) is:

$$\log C_{ts} = 5.435 - 0.286\log\left(\frac{\omega d}{\nu}\right) - 0.457\log\left(\frac{U_\star}{\omega}\right) + \left[1.799 - 0.409\log\left(\frac{\omega d}{\nu}\right) - 0.314\log\left(\frac{U_\star}{\omega}\right)\right] \times \log\left(\frac{US_0}{\omega} - \frac{V_{cr}S_0}{\omega}\right) \tag{39}$$

where C_{ts} is the total sediment transport concentration (in ppm by weight); ω is the terminal velocity; d is the median sieve diameter of sediment particles; ν is the kinematic viscosity, $U_\star = \sqrt{gHS_0}$ is the shear velocity; V_{cr} is the critical velocity at incipient motion, and the logarithms are for base 10. Rewriting this equation to replace U_* and U with the expression for the shear velocity and the volumetric flow rate terms, followed by rearrangement, gives:

$$\frac{\log C_{ts} - 5.435 - 0.286\log\left(\frac{\omega d}{\nu}\right) - 0.457\log\left(\frac{\sqrt{gHS_0}}{\omega}\right)}{1.799 - 0.409\log\left(\frac{\omega d}{\nu}\right) - 0.314\log\left(\frac{\sqrt{gHS_0}}{\omega}\right)} = \log\left[S_0\left(\frac{Q}{\omega HW} - \frac{V_{cr}}{\omega}\right)\right] \tag{40}$$

When the parameters are specified, this equation is a relation between H and S_0. The numerical coefficients in this equation are given to three or four significant figures, which limits the precision of any calculations.

Yang (1996) calculated the values of S_0 for different water depths H using Eq. (40) for the case of the Rio Grande river and provided the results for slope and power that appear here in the first two columns of Table 1 as S_{0Y} and US_{0Y}. These values are plotted in Fig. 1. Yang (1996) let H vary between 1.71 and 3.51 ft. For our calculations we use a range of 1.50–3.51 ft. Our results are in columns 3 and 4 of the table, denoted as S_{0GGK} and US_{0GGK} and are plotted in red in Fig. 1. Observe that for both sets of data, the number of significant figures presented is more than can be justified based on Eq. (40). The discrepancy between the two sets of values is small. The difference can be attributed to the fact that Yang (1996) did not provide values of V_{cr}/ω and ω, so these had to be estimated, with values selected to be 2.5 and 0.43 m s^{-1}, respectively. We also assumed $\nu = 1 \times 10^{-6}$ m^2 s^{-1} and $g = 9.81$ m s^{-2}. Data values common to both sets of calculations are $C_{ts} = 517$ ppm, $d = 0.31$ mm, $W = 370$ ft, and $Q = 2877$ cfs. To compare the datasets, we performed a single-factor ANOVA analysis using the nine values of slope (multiplied by 10^4) from the Y and GGK results associated with common values of H. From this analysis, we found that $F_{\text{crit}} = 4.49$ while $F = 2.63 \times 10^{-4}$. Thus there is no statistically significant difference between the two datasets.

Table 1 Values of slope and power reported by Yang (1996) and calculated here using estimated values of V_{cr}/ω and ω.

H (ft)	$S_{0_Y} \times 10^4$	$US_{0_Y} \times 10^3$ (ft/s)	$S_{0_{GGK}} \times 10^4$	$US_{0_{GGK}} \times 10^3$ (ft/s)
3.51	11.14	2.451	11.321	2.4825
3.08	9.77	2.443	9.833	2.4573
2.75	8.70	2.435	8.722	2.4410
2.49	7.84	2.431	7.863	2.4305
2.27	7.15	2.430	7.147	2.4233
2.08	6.57	2.432	6.536	2.4185
1.93	6.08	2.433	6.058	2.4158
1.79	5.66	2.434	5.615	2.4142
1.71	5.41	2.435	5.363	2.4138
1.65	–	–	5.175	2.4137
1.60	–	–	5.018	2.4138
1.50	–	–	4.706	2.4145

We performed a least square linear fit to the datasets and found that for Yang's data:

$$S_{0_Y} \times 10^4 = 3.189H - 0.0650 \tag{41}$$

with an R^2 of 0.9999. For the full calculated GGK dataset, the best fit linear equation is:

$$S_{0_{GGK}} \times 10^4 = 3.275H - 0.2483 \tag{42}$$

with $R^2 = 0.9997$. Both datasets are thus essentially linear. Furthermore, if we force a best fit of the data with a straight line passing through the origin, the slopes of the Y and GGK data change only slightly to 3.162 and 3.170 respectively while the values of R^2 are unchanged. These results confirm that, with either dataset, $S_0 = C_1H$ is a very good fit. The complicated expression in Eq. (40) is essentially a linear relation between S_0 and H.

So, what is the consequence of knowing that $S_0 = C_1H$ where C_1 is a constant? This question can be answered by substituting this linear relation into the definition of stream power yielding:

$$P = US_0 = \frac{QS_0}{WH} = \frac{QC_1}{W} = C_2 \tag{43}$$

where C_2 is a constant. This shows that the stream power is a constant function of H and S_0. Indeed, the results in Yang (1976, 1996) show that the calculated stream power as function of water depth is constant up to the third significant figure! In fact, Yang (1976, 1994) needed the fourth figure to claim that US_0 has a minimum. Given measurement, round off and modelling errors, a more reasonable conclusion is that the minimization of stream power leads to a linear sediment transport relation $S_0 = C_1H$. Such a linear relationship is consistent with Eq. (40), providing some hope for the minimization

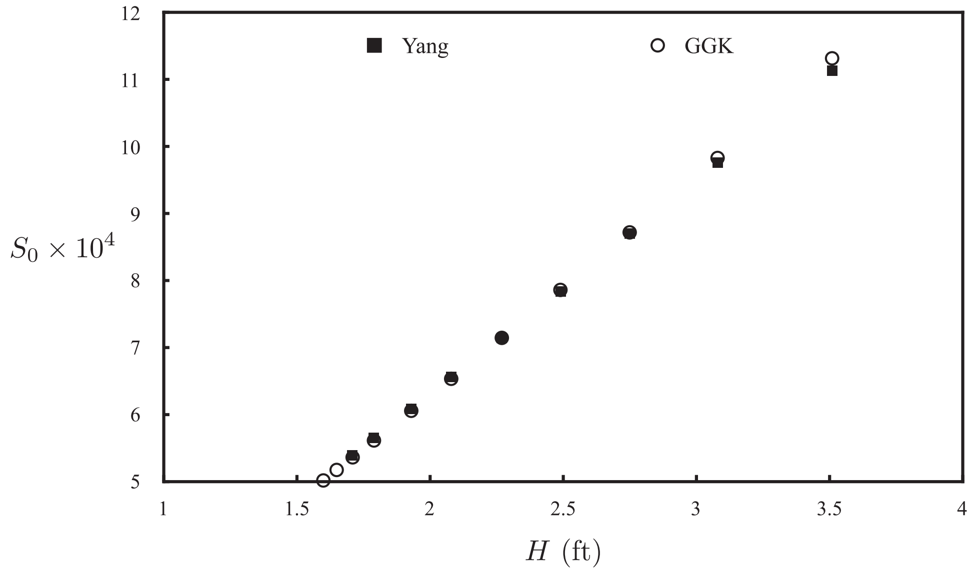

Figure 1 Plot of S_0 vs. H obtained from Eq. (40) in Yang (1996) (black squares and first column of Table 1) and calculations here (open circles and third column of Table 1)

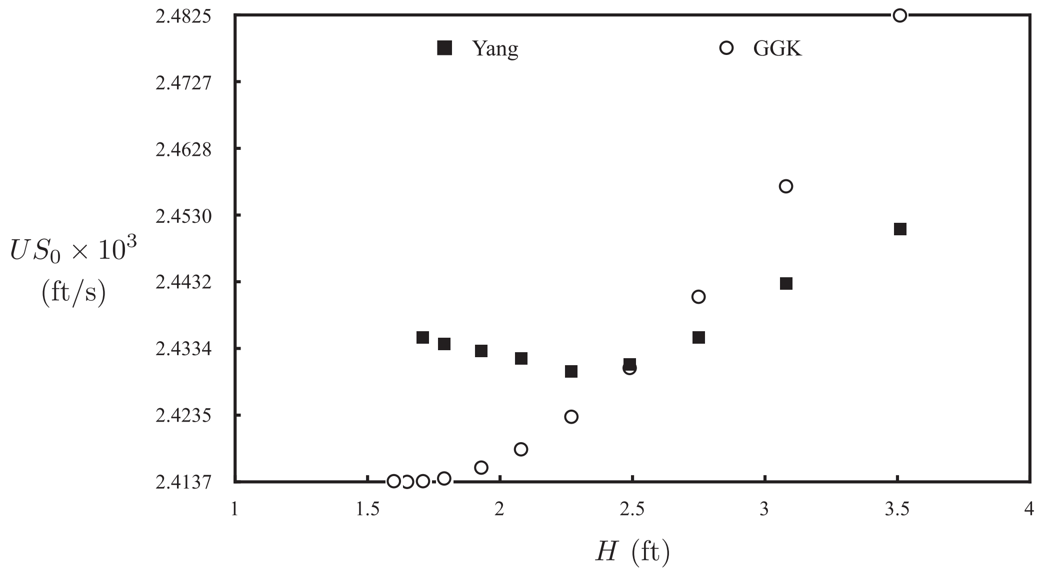

Figure 2 Plot of US_0 vs. H obtained from values of S_0 computed from Eq. (40) in Yang (1996) (black squares and second column of Table 1) and calculations here (open circles and fourth column of Table 1)

principle. On the other hand, $S_0 = C_1 H$ also leads to constant stream power and constant entropy as a function of water depth and slope. That is, US_0 is not convex and no minimum exists. Therefore, the minimization problem has no unique solution.

As further confirmation of this conclusion, we plotted the values of US_{0Y} and the values of US_{0GGK} as appears in Fig. 2. Note that Yang's data has a minimum, based on the fourth digit, which he takes to be the solution of the mEPP problem. Our data only goes through a minimum at a value of H outside the range Yang considered. Again, we performed an ANOVA single factor analysis of the $US_0 \times 10^3$ datasets. We found that $F_{\text{crit}} = 4.49$ while $F = 0.136$. Thus there is no statistical difference between the US_{0Y} and US_{0GGK} datasets. This means there is no basis for claiming that the fact that one set goes through a minimum (even at the fourth figure) while the other does not is significant and can support a theory.

4 The mEPP for a channel with rigid walls

As an additional convenient step in examining the mEPP for an open channel, consider the case when the density of the fluid is constant and the walls of the channel are rigid. Clearly, Eq. (29) cannot be directly assessed for minimum entropy generation based on an extremum principle. This expression consists of only one flux, S_f or S_f/H, and one force, Q; and thus there is no quantity to be minimized. Furthermore, the mEPP relies on the flux being linearly proportional to the force (Kay, 2003;

Trepczyńska-Lent, 2012). This would require an expression such as:

$$\frac{S_f}{H} = LQ \tag{44}$$

where L is a constant of proportionality. However the usual form of the friction slope, valid for steady flow and for unsteady flows, provided that the time scale of the wave is longer than the time scale of turbulence diffusion, is of the form:

$$S_f = \frac{kQ^2}{H^m} \tag{45}$$

For example, for the Manning's formula, $k = n^2$ and $m = 10/3$ while for the Chezy formula, $k = 1/C^2$ and $m = 3$ where n is Manning's n and C is the Chezy coefficient. Then if the normal flow is characterized as having a depth H_N with:

$$S_0 = \frac{kQ^2}{H_N^m} \tag{46}$$

we obtain:

$$S_f = S_0 \left(\frac{H_N}{H}\right)^m \tag{47}$$

Substitution of Eq. (47) into Eq. (29) gives the expression for entropy generation as:

$$\frac{1}{\overline{\theta}} \overline{\rho} g Q \frac{S_0}{H_N} \left(\frac{H_N}{H}\right)^{m+1} = \overline{\Lambda} \tag{48}$$

This result indicates that the entropy generation rate will be minimized only when the flow is as deep as possible. This makes physical sense because this would minimize the velocity, thereby minimizing energy losses due to friction. Indeed for a fixed Q, S_f is smallest when the velocity is low and the depth is large. Although this observation indicates that entropy generation is minimized when S_0 is near zero so that H_N is large for a constant Q, this conclusion has little relevance for the mEPP and the conditions under which it is applied. Furthermore, the conclusion does not require that the system be at steady state. At steady state, based on Eq. (48), when $H > H_N$, the entropy production rate for gradually varied flow will be less than for normal flow. This corresponds to M1, S1 and S2 profiles. On the other hand, when $H_N > H$ for a steady flow, the entropy production will be greater for the gradually varied flow than for the normal flow with the same flow rate. This corresponds to M2, M3 and S3 profiles.

To determine the entropy production rate in light of a constitutive form for S_f, we can substitute Eq. (45) into Eq. (29) to obtain the entropy production rate in the channel as:

$$\frac{1}{\overline{\theta}} \left(\overline{\rho} g \frac{kQ^3}{H^{m+1}}\right) = \overline{\Lambda} \tag{49}$$

Now consider the flow in the channel after the slope has adjusted so that the depth of flow at the inlet boundary is the normal depth. If this depth is denoted H_1 and the entropy generation rate per volume is $\overline{\Lambda}_1$, then the entropy generation rate per unit volume is:

$$\frac{1}{\overline{\theta}} \left(\overline{\rho} g \frac{kQ^3}{H_1^{m+1}}\right) = \overline{\Lambda}_1 \tag{50}$$

The ratio of these two entropy production rates is:

$$\frac{\overline{\Lambda}}{\overline{\Lambda}_1} = \left(\frac{H_1}{H}\right)^{m+1} \tag{51}$$

This equation means that if the depth of inflow to a channel is greater than the normal depth, the entropy production rate per unit volume will be less than the production rate for normal flow. On the other hand, if the inlet depth is less than the normal depth, the entropy production rate will be greater than that for normal flow. Thus, for a fixed flow rate and depth of inflow, an adjustment of the slope of the stream bed to a state where the flow will be normal does not mean that the entropy production rate is being minimized or even reduced. These observations are consistent with the result that entropy production is smallest when the depth is greatest. Thus, the use of an energy minimization principle with open channel flow is not physically warranted.

5 Other examples of mEPP failures

That the minimum entropy theory does not hold for open channel flow is by no means the first instance that this theory has been found to fail. In fact, there are a broad range of manuscripts devoted to problems where mEPP has been found to be inapplicable. Here, we comment on several of the common ones.

Some manuscripts purporting to examine the mEPP look at simple systems that can be analysed both on the basis of mechanistic equations and in light of an effort to minimize entropy production. For example, Herrmann (1986) examines the voltage distribution and current when a pair of resistors are connected in series and in parallel. The paper claims that the entropy production is minimum at the steady state. However, Landauer (1975) finds that even for the case of linear circuits, the minimal entropy production theorem does not apply. Martyushev, Nazarova, and Seleznev (2007) disagree with a result of Jaynes (1980) which involves conduction through resistors at two different temperatures, noting that there need to be at least two different flows. Since the study of Jaynes involves only a single flow of current that is distributed between two resistors, the example is actually not one that could be studied using the mEPP.

Bertola and Cafaro (2008) and Palffy-Muhoray (2001) investigated one-dimensional heat flow in a solid. They assert that although the steady-state temperature solution for constant thermal conductivity is a linear profile, the minimum entropy production occurs for an exponential profile. Although this seems to suggest a contradiction, the fact that the exponential profile does not correspond to a steady state solution means that it does not correspond to one of the conditions for the mEPP theorem (Hoover, 2002). Zullo (2016) also considers heat conduction and concludes that the breakdown of the mEPP for such problems lies in the fact that the phenomenological coefficient is not constant but depends on temperature. Sahin (2011) concludes that for steady state heat conduction, the entropy generation rate at steady state is minimal only when the temperature gradient is zero such that there is no heat transfer. In a related vein, Barragán (2009) shows that the steady states described by Newton's law are not states having minimum entropy production. Based on these studies, it is reasonable to assert, as did Kay (2003), that the mEPP does not apply unless a system is isothermal.

Martyushev et al. (2007) note that the essence of the mEPP is that free (unfixed) thermodynamic forces in a system are mutually adjusted to bring the system to the state with a minimum entropy dissipation. If only one force is present and it is not zero, with the zero case equating to a system of thermodynamic equilibrium with no entropy production, the variation used in the proof vanishes and further discussion about making entropy production an extreme is pointless. The mEPP has sense only if several forces are available and some of them are fixed. The neglect of this observation and attempts to go beyond the scope of the theorem lead to erroneous results, which invalidate the essence of the theorem. The mEPP has been discussed, generalized, employed, and criticized by different researchers (e.g.De Groot & Mazur, 1984; Gyarmati, 1969, 1970; Mamedov, 2003; Reis, 2014; Veveakis & Regenauer-Lieb, 2015; Ziegler, 1963). Auxiliary to this discussion is the question of whether the mEPP is useful and capable of providing information beyond that obtained from analysis of the conservation equations at steady state. Some researchers have expressed their scepticism (e.g. Jaynes, 1980; Klein, 1960; Martyushev et al. 2007). The problem that arises is that the information needed to employ the mEPP must be so extensive that nothing new is added by including the mEPP (Martyushev, 2013).

In summary, the statements and applications of the mEPP are criticized because it has not been applied properly, may only hold at equilibrium, and may not really add any new information to an analysis. These criticisms also apply when attempting to use the mEPP to analyse open channel flow.

6 Conclusion

We have analysed application of Prigogine's minimum entropy production principle to the case of one-dimensional flow in both fluvial and non-fluvial channels. The conditions under which this principle may be applied to any physical system have been provided. Analysis of the entropy inequality, subject to constraints provided by thermodynamic principles and the conservation equations of mass, momentum, and total energy demonstrates that the resulting expression for the rate of entropy production cannot be assessed making use of the mEPP.

Minimization of the unit stream power of a sedimentary channel subject to the same conditions imposed by proponents of the minimum stream power theory leads to a linear relation between channel slope and water depth which in turn leads to a unit stream power that is constant for all slopes and depths. These relations agree with published minimum unit stream power data up to the available significant figures. The only logical conclusion is that the stream power is a constant to within these significant figures (i.e. no unique minimum exists). Proponents of the minimum stream power formalism have been basing their conclusions on digits beyond those which are significant with total disregard for the fact that their measured parameters and their empirical relations do not allow for such accuracy.

Parker (1977) concludes his discussion of Yang (1976) by stating

> The writer would not recommend the use of the Yang resistance relation in its present form in practical applications. However, serious attention should be paid to the author's concept of minimum stream power which appears capable of producing a reasonable resistance relation.

Due consideration to this theory is given in the present paper which shows that it is not capable of producing a reasonable resistance relation systematically.

Further analysis for the case of non-sedimentary channels shows that for a fixed flow rate, when the depth is greater than the normal depth, the entropy production rate is less than for normal flow. Additionally, when the depth of flow is less than the normal depth, adjustment of the channel slope so that the flow is uniform and normal reduces the entropy production rate but not to a minimum. In summary, Prigogine's mEPP does not apply to the study of open channel flow.

It is found that the entropy generation rate, and thus the unit stream power, is minimized when the flow is as deep as possible. This conclusion is valid for steady and unsteady as well as uniform and non-uniform flows. For the steady state case, the analysis reveals that the entropy production (i.e. unit stream power) for a non-uniform flow is (i) larger than the entropy production in uniform flow for M2, M3 and S3 profiles, but (ii) smaller than the entropy production in a uniform flow for M1, S1 and S2 profiles. Therefore, requiring that the unit stream power be extremum is at best ad-hoc and beyond what the conservation laws demand.

Funding

The first author acknowledges support provided by 10.13039/100000183 Army Research Office grant W911NF-14-1-02877, US Department of Energy grant DE-SC0002163, and 10.13039/100000001 National Science Foundation grant 1619767. The second author acknowledges the financial support by the Research Grants Council (RGC) of Hong Kong (grants 612713 and T21-602/15R).

Notation

ANOVA = analysis of variance
$\boldsymbol{a}$ = vector of system parameters
b = entropy source per unit volume per unit time
C_1 = constant of integration
C_2 = scaled value of constant C_1
C_{ts} = total sediment concentration
d = median sieve diameter of sediment particles
E = internal energy per volume at a point
$\overline{E}$ = internal energy per volume at a point averaged over the cross section and, if desired, time
$\boldsymbol{g}$ = gravitational vector
F = statistical variable for two datasets which indicates datasets are statistically equivalent when $F < F_{\text{crit}}$
F_{crit} = statistical variable used to determine if datasets are different
f = empirical function of Q, H, and system parameters proposed to be equal to S_0
g = magnitude of gravitational vector
H = depth of flow
H_N = depth of normal flow
h = body source of energy per unit volume per unit time
k = coefficient
L = coefficient of proportionality
m = exponent in friction slope equation
P = stream power
p = pressure
$\overline{p}$ = averaged pressure
Q = total volumetric flow rate
Q^s = volumetric flow rate of sediment
Q^w = volumetric flow rate of water
$\boldsymbol{q}$ = non-advective point thermal energy flux vector
$\overline{\overline{q}}_x$ = averaged component of the non-advective thermal energy flux vector, including contributions from turbulence and energy non-uniformity over the averaging region
R^2 = statistical measure of correlation between data and a regression line
S_0 = channel slope
S_{0Y} = channel slope calculated based on Yang's data
S_{0GKK} = channel slope calculated here based on full dataset
S_f = friction slope
$\mathbf{t}$ = total stress tensor at a point in the fluid
t_{xx} = component of the stress tensor equal to $\boldsymbol{i} \cdot \mathbf{t} \cdot \boldsymbol{i}$ where $\boldsymbol{i}$ is the unit vector in the direction of flow
$\overline{\overline{t}}_{xx}$ = averaged component of the stress tensor acting in the x direction on a cross-sectional face of the channel; includes turbulent components and those associated with spatial variability of velocity over the cross section
t = time
U = density weighted average velocity along the channel
U_* = shear velocity
V_{cr} = flow velocity at incipient motion of sediment particles
$\boldsymbol{v}$ = point velocity vector
W = channel width
x = coordinate direction along the channel
η = point value of entropy per volume
$\overline{\eta}$ = averaged entropy per unit volume
θ = point value of temperature
$\overline{\theta}$ = entropy weighted average temperature
Λ = point rate of entropy production per volume
$\overline{\Lambda}$ = averaged rate of of entropy production per volume at a cross-section
$\overline{\Lambda}_1$ = rate of energy production per unit volume when the inlet depth of flow is the normal depth
λ = Lagrange multiplier subscripted to indicate association with a particular equation
μ = chemical potential at a point
ν = kinematic viscosity
$\overline{\mu}$ = density weighted chemical potential
ρ = fluid mass density at a point
$\overline{\rho}$ = average fluid mass density
$\boldsymbol{\varphi}$ = non-advective entropy flux vector
$\overline{\overline{\varphi}}_x$ = averaged component of non-advective entropy flux in the direction of flow which also accounts for contributions due to non-uniform entropy distribution over the surface and turbulence
ω = terminal velocity

References

Bailyn, M. (1994). *A survey of thermodynamics* . New York, NY: American Institute of Physics Press.

Barragán, D. (2009). Producción de entropía y ley de enfriamiento de newton. *Revista Ingeniería e Investigación*, *29*(2), 88–93.

Bertola, V., & Cafaro, E. (2008). A critical analysis of the minimum entropy production theorem and its application to heat and fluid flow. *International Journal of Heat and Mass Transfer*, *51*, 1907–1912.

Callen, H. B. (1985). *Thermodynamics and an introduction to thermostatistics*. New York, NY: Wiley.

De Groot, S. R., & Mazur, P. (1984). *Non-equilibrium thermodynamics*. New York, NY: Dover.

Eringen, A. C. (1980). *Mechanics of continua*. Huntington, NY: Krieger.

Gray, W. G. (2002). On the definition and derivatives of macroscale energy for the description of multiphase systems. *Advances in Water Resources*, *25*(8–12), 1091–1104.

Gray, W. G., & Ghidaoui, M. S. (2009). Thermodynamic analysis of stream flow hydrodynamics. *Journal of Hydraulic Research*, *47*(4), 403–417. doi:10.3826/jhr.2009.3331

Gray, W. G., Leijnse, A., Kolar, R. L., & Blain, C. A. (1993). *Mathematical tools for changing spatial scales in the analysis of physical systems*. Boca Raton, FL: CRC Press.

Gray, W. G., & Miller, C. T. (2009). Thermodynamically constrained averaging theory approach for modeling flow and transport phenomena in porous medium systems: 7. Single-phase megascale flow models. *Advances in Water Resources*, *32*(8), 1121–1142. doi:10.1016/j.advwatres.2009.05.010

Gray, W. G., & Miller, C. T. (2011). On the algebraic and differential forms of Darcy's equation. *Journal of Porous Media*, *14*(1), 33–50.

Gray, W. G., & Miller, C. T. (2013). A generalization of averaging theorems for porous medium analysis. *Advances in Water Resources*, *62*, 227–237. doi:10.1016/j.advwatres.2013.06.006

Gyarmati, I. (1969). On the "governing principle of dissipative processes" and its extension to non-linear problem. *Annalen der Physik*, *478*, 353–378.

Gyarmati, I. (1970). *Non-equilibrium thermodynamics*. Berlin: Springer.

Herrmann, F. (1986). Simple examples of the theorem of minimum entropy production. *European Journal of Physics*, *7*, 130–131.

Hoover, W. G. (2002). Note on "Comment on 'A check of Prigogine's theorem of minimum entropy production in a rod in a nonequilibrium stationary state' by Irena Danielewicz-Ferchmin and A. Ryszard Ferchmin" [Am. J. Phys. 68 (10), 962–965 (2000)], by Peter Palffy-Muhoray [Am. J. Phys. 69 (7), 962–965 (2001)]. *American Journal of Physics*, *70*(4), 452–454.

Jaynes, E. T. (1980). The minimum entropy production principle. *Annual Review of Physical Chemistry*, *31*, 579–601.

Jou, D., & Casas-Vázquez, J. (2001). Extended irreversible thermodynamics and its relation with other continuum approaches. *Journal of Non-Newtonian Fluid Mechanics*, *96*(1–2), 77–104.

Kay, J. J. (2003). *About some common slipups in applying Prigogine's minimum entropy production principle to living systems*. Retrieved from http://www.jameskay.ca/musings/mep.pdf (Online only)

Klein, M. J. (1960). The principle of minimum entropy production. *Rendiconti della Scuola Internazionale di Ficica "Enrico Fermi"*, *10*, 198–204.

Landauer, R. (1975). Inadequacy of entropy and entropy derivatives in characterizing the steady state. *Physical Review A*, *12*(2), 636–638.

Mamedov, M. M. (2003). On the incorrectness of the traditional proof of the Prigogine principle of minimum entropy production. *Technical Physics Letters*, *29*, 340–341.

Martyushev, L. M. (2013). Entropy and entropy production: Old misconceptions and new breakthroughs. *Entropy*, *15*, 1152–1170.

Martyushev, L. M., Nazarova, A. S., & Seleznev, V. D. (2007). On the problem of the minimum entropy production in the nonequilibrium stationary state. *Journal of Physics A: Mathematical and Theoretical*, *40*, 371–380.

Palffy-Muhoray, P. (2001). Comment on "A check of Prigogine's theorem of minimum entropy production in a rod in a nonequilibrium stationary state" by Irena Danielewicz-Ferchmin andA. Ryszard Ferchmin [Am. J. Phys. 68 (10), 962–965 (2000)]. *American Journal of Physics*, *69*(7), 825–826.

Parker, G. (1977). Discussion of "Minimum unit stream power and fluvial hydraulics". *Journal of the Hydraulics Division, ASCE*, *103*(HY7), 811–816.

Reis, A. H. (2014). Use and validity of principles of extremum of entropy production in the study of complex systems. *Annals of Physics*, *346*, 22–27.

Sahin, A. Z. (2011). Entropy production minimization in steady state heat conduction. *International Journal of the Physical Sciences*, *6*(12), 2826–2831.

Trepczyńska-Lent, M. (2012). The criterion of minimum entropy production in eutectic growth. *Archives of Foundry Engineering*, *12*(2), 85–88.

Truesdell, C. (1984). *Rational thermodynamics*. Berlin: Springer.

Veveakis, E., & Regenauer-Lieb, K. (2015). Review of extremum postulates. *Current Opinion in Chemical Engineering*, *7*, 40–46.

Yang, C. T. (1976). Minimum unit stream power and fluvial hydraulics. *Journal of the Hydraulics Division, ASCE*, *102*(HY7), 919–934.

Yang, C. T. (1994). Variational theories in hydrodynamics and hydraulics. *Journal of Hydraulic Engineering*, *120*(6), 737–756.

Yang, C. T. (1996). *Sediment transport theory and practice*. New York, NY: McGraw-Hill.

Ziegler, H (1963). Some extremum principles in irreversible thermodynamics with application to continuum mechanics. In I. N. Sneddon & R. Hill (Eds.), *Progress in solid mechanics* (Vol. 4, 91 pp). Amsterdam: North Holland.

Zullo, F. (2016). Entropy production in the theory of heat conduction in solids. *Entropy*, *18*(3), 87. doi:10.3390/e18030087