Multi-Leak Detection For Water Supply Systems: A Compressive Sensing Perspective

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Outline

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System Model

System Profiling

System Profiling

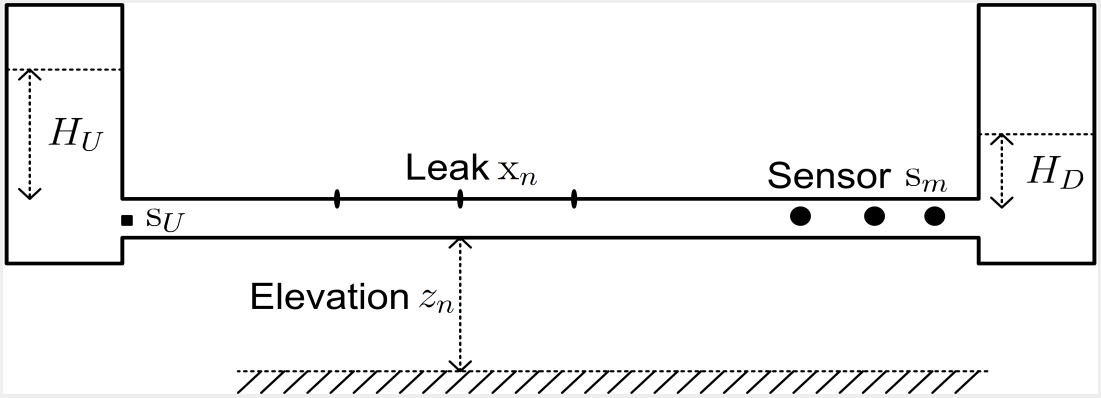


Figure: The water supply system model with multiple transducers.

- x_n, the nth leak position, $\forall n = 1 : N$.
- u_n, the size of the nth leak.
- s_m, the mth transducer position, $\forall m = 1 : M$.
- s_U, the upstream node position.
- z_n, the elevation of the nth leak.
- $\mathrm{s}_U < \mathrm{x}_1 < \cdots < \mathrm{x}_N < \mathrm{s}_1 < \cdots < \mathrm{s}_M$.

System Model

General Model

$\mathbf{y}_m(\omega) = \begin{bmatrix} q(\mathrm{s}_m;\omega) \\ h(\mathrm{s}_m;\omega) \end{bmatrix}$ is the vector of discharge $q(\mathrm{s}_m;\omega)$ and head $h(\mathrm{s}_m;\omega)$ response at the mth transducer, $\forall\omega = 1:\Omega$, $\forall m = 1:M$, and $\mathbf{y}_U(\omega) = \begin{bmatrix} q(\mathrm{s}_U;\omega) \\ h(\mathrm{s}_U;\omega) \end{bmatrix}$.

General Model

Then the measurement function is given by (which is nonlinear w.r.t. u_n and x_n)

$$\mathbf{y}_m(\omega) = \mathbf{F}(\mathrm{s}_m - \mathrm{x}_N;\omega) \prod_{n=N:1} \mathbf{V}(\mathrm{u}_n)\mathbf{F}(\mathrm{x}_n - \mathrm{x}_{n-1};\omega)\mathbf{y}_U(\omega) + \epsilon_m(\omega), \tag{1}$$

where

$$\mathbf{F}(x;\omega) = \begin{bmatrix} \cosh(\mu(\omega)x) & -\dfrac{\sinh(\mu(\omega)x)}{Z(\omega)} \\ -Z(\omega)\sinh(\mu(\omega)x) & \cosh(\mu(\omega)x) \end{bmatrix}, \tag{2}$$

$$\mathbf{V}(\mathrm{u}_n) = \begin{bmatrix} 1 & -\mathrm{v}_n(\mathrm{u}_n) \\ 0 & 1 \end{bmatrix}, \quad \mathrm{v}_n(\mathrm{u}_n) = \left(\frac{g}{2(H_n - z_n)}\right)^{\frac{1}{2}} \mathrm{u}_n. \tag{3}$$

System Model

Approximately Linear Model

Approximately Linear Model

For the small leak size, *i.e.*, $\mathrm{u}_n \ll 1$, $\forall n = 1:N$, the measurement model in Eq. (1) can be approximated using a linear form w.r.t. $\mathrm{v}_n(\mathrm{u}_n)$ [Wang X. et.al. 2017]

$$\mathbf{y}_m(\omega) = \mathbf{\Phi}(\mathrm{s}_m;\omega)\mathbf{y}_U(\omega) + \sum_{n=1:N} \mathbf{\Psi}(\mathrm{s}_m,\mathrm{x}_n;\omega)\mathbf{y}_U(\omega)\mathrm{v}_n(\mathrm{u}_n) + \boldsymbol{\epsilon}_m^{\flat}(\omega) + \boldsymbol{\epsilon}_m(\omega), \tag{4}$$

where $\boldsymbol{\epsilon}_m^{\flat}(\omega)$ is the approximation error in the order of $o\Big(\max_{n=1:N}\{\mathrm{v}_n(\mathrm{u}_n)\}\Big)$,

$$\mathbf{\Phi}(\mathrm{s}_m;\omega) = \begin{bmatrix} \cosh\left(\mu(\omega)\mathrm{s}_m\right) & -\dfrac{\sinh\left(\mu(\omega)\mathrm{s}_m\right)}{Z(\omega)} \\ -Z(\omega)\sinh\left(\mu(\omega)\mathrm{s}_m\right) & \cosh\left(\mu(\omega)\mathrm{s}_m\right) \end{bmatrix}, \tag{5}$$

$$\mathbf{\Psi}(\mathrm{s}_m,\mathrm{x}_n) = \begin{bmatrix} Z\cosh\left(\mu(\mathrm{s}_m-\mathrm{x}_n)\right)\sinh\left(\mu\mathrm{x}_n\right) & -\cosh\left(\mu(\mathrm{s}_m-\mathrm{x}_n)\right)\cosh\left(\mu\mathrm{x}_n\right) \\ -Z^2\sinh\left(\mu(\mathrm{s}_m-\mathrm{x}_n)\right)\sinh\left(\mu\mathrm{x}_n\right) & Z\sinh\left(\mu(\mathrm{s}_m-\mathrm{x}_n)\right)\cosh\left(\mu\mathrm{x}_n\right) \end{bmatrix}, \tag{6}$$

System Model

Compact Linear Model

In practice, only the head response is available due to the limit of transducer functions. Let's define the head difference $\mathrm{z}_m(\omega)$ and vector $\psi_m(\mathrm{x}_n;\omega)$ below,

$$\mathrm{z}_m(\omega) = h_m(\omega) - \begin{bmatrix} -Z(\omega)\sinh\left(\mu(\omega)\mathrm{s}_m\right) \\ \cosh\left(\mu(\omega)\mathrm{s}_m\right) \end{bmatrix}^\top \mathbf{y}_U(\omega), \tag{7}$$

$$\psi_m(\mathrm{x}_n;\omega) = \begin{bmatrix} -(Z(\omega))^2\sinh\left(\mu(\omega)(\mathrm{s}_m-\mathrm{x}_n)\right)\sinh\left(\mu(\omega)\mathrm{x}_n\right) \\ Z(\omega)\sinh\left(\mu(\omega)(\mathrm{s}_m-\mathrm{x}_n)\right)\cosh\left(\mu(\omega)\mathrm{x}_n\right) \end{bmatrix}^\top \mathbf{y}_U(\omega). \tag{8}$$

The Compact-Form of Linear Model

The head difference $\mathrm{z}_m(\omega)$ associated with a certain angular frequency ω, $\forall\omega = 1:\Omega$, at the sensor node s_m, $\forall m = 1:M$, is cast as a linear function of leak size $\mathrm{v}_n(\mathrm{u}_n)$,

$$\mathrm{z}_m(\omega) = \sum_{n=1:N} \psi_m(\mathrm{x}_n;\omega)\mathrm{v}_n(\mathrm{u}_n) + \tilde{\epsilon}_m(\omega), \tag{9}$$

where $\tilde{\epsilon}_m(\omega) = \epsilon_m(\omega) + \epsilon_m^\flat(\omega)$ is the equivalent measurement error.

$$\mathbf{z} = \sum_{n=1:N} \boldsymbol{\psi}(\mathrm{x}_n)\mathrm{v}_n + \tilde{\boldsymbol{\epsilon}}(\omega), \tag{10}$$

Outline

Sparseness Structure

Theoretical Evidence

Theoretical Evidence

- The number of leaks is always finite in practice.
- The measurement function in Eq.(9) is a linear form w.r.t. v_n.
- The spatial-domain correlation of the base vectors is moderate.

$$\varrho(\Delta\chi_k;\chi_k) \leq \varepsilon,\ \forall \Delta\chi_k \geq \delta,\ \forall \chi_k \in [\mathrm{s}_U, \mathrm{s}_D], \tag{11}$$

$$\varrho(\Delta\chi_k;\chi_k) \leq \varepsilon,\ \forall \Delta\chi_k \geq \delta,\ \forall \chi_k \in [\mathrm{s}_U, \mathrm{s}_D]. \tag{12}$$

A moderate spatial-domain correlation of the base vectors leads to an effective (error-bounded and sparseness-sufficient) compressive sensing model for leak detection.

Definition: Spatial-Domain Correlation

Given two position samples χ_k and $\chi_k + \Delta\chi_k$ with a certain difference $\Delta\chi_k$, the spatial-domain correlation of base vector $\phi(\chi_k)$ is defined as

$$\varrho(\Delta\chi_k;\chi_k) = \left| \frac{(\phi(\chi_k))^\top \phi(\chi_k + \Delta\chi_k)}{\|\phi(\chi_k)\|_2} \right|. \tag{13}$$

Sparseness Structure

Exploiting Sparseness Nature

Exploiting Sparseness Nature

- Draw a set of candidate location samples $\{\chi_k | \forall k = 1 : K\}$ uniformly, $\chi_k \in [s_U, s_D]$, and K is supposed as the number of location samples.
- Construct a base vector $\phi(\chi_k) = \mathrm{vec}[\psi_m(\chi_k; \omega) | \forall \omega = 1 : \Omega, \forall m = 1 : M]$.
- Construct a dictionary $\mathbf{F}(\boldsymbol{\chi}) = \left(\mathrm{mat}[\phi^\top(\chi_k) | \forall k = 1 : K]\right)^\top$.
- Then, the measurement vector $\mathbf{z}$ can be cast as

$$\mathbf{z} = \sum_{k=1:K} \phi(\chi_k) \mathrm{u}'_k + \varsigma, \tag{14}$$

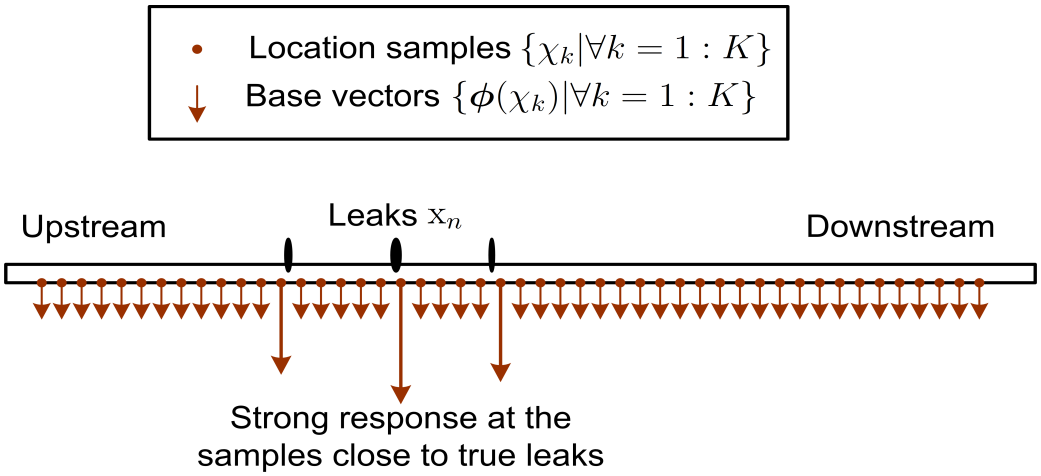
$$\mathbf{z} = \mathbf{F}(\boldsymbol{\chi})\mathbf{u} + \varsigma, \tag{15}$$

where $\mathbf{z} = \mathrm{vec}[\mathrm{z}_m(\omega) | \forall \omega = 1 : \Omega, \forall m = 1 : M]$, $\mathbf{u}$ is the sparse coefficient vector, and $\varsigma = \epsilon + \epsilon^\flat + \epsilon_s$ is the equivalent measurement error, in which $\epsilon^\flat$ is the linear modeling error, and ϵ_s is the sparse modeling error.

- A sparse signal means most of its elements are zero.

Sparseness Structure

Illustration



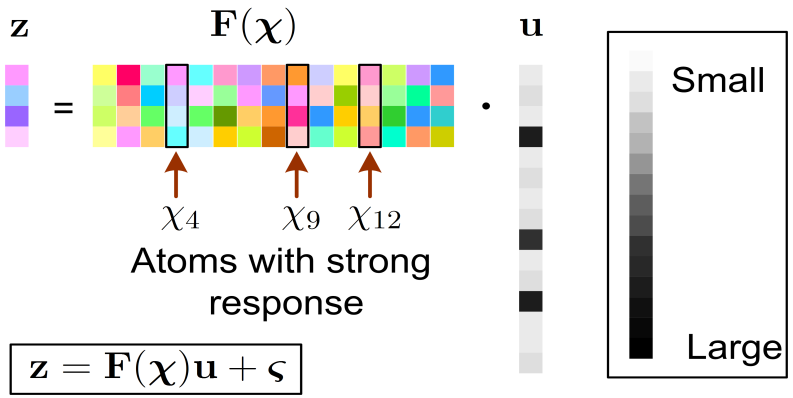


Figure: Illustration of the sparseness nature: $\mathbf{z} = \mathbf{F}(\boldsymbol{\chi})\mathbf{u} + \boldsymbol{\varsigma}$.

Outline

Compressive Sensing-Based Detection

3.1 Problem Formulation

3.1 Problem Formulation: The Ultimate Goal

The goal of multiple-leak detection is to estimate the positions and sizes of all leaks *i.e.*, $\{\mathrm{x}_n, \mathrm{v}_n | \forall n = 1 : N\}$, given $\{h_m(\omega), \mathrm{s}_m | \forall \omega = 1 : \Omega, \forall m = 1 : M\}$.

3.1.1(1) General Solution: Statistically Optimal Approach)

The maximum likelihood estimate is formulated as, based on the general model,

$$\mathcal{P}_1 : \ \hat{\mathbf{x}}, \hat{\mathbf{v}} = \arg\max_{\mathbf{x},\mathbf{v}} \prod_{\substack{\omega=1:\Omega,\\ m=1:M}} \mathcal{N}\big(h_m(\omega) | r_m(\mathbf{x}, \mathbf{v}; \omega), \mathrm{w}(\omega)\big), \tag{16}$$

where $r_m(\mathbf{x}, \mathbf{v}; \omega) = [\mathbf{R}_m(\mathbf{x}, \mathbf{v}; \omega)\mathbf{y}_U(\omega)]_2$, $[\cdot]_2$ denotes the second element of a vector, and $\mathbf{R}_m(\mathbf{x}, \mathbf{v}; \omega) = \mathbf{F}(\mathrm{s}_m - \mathrm{x}_N; \omega) \prod_{n=N:1} \mathbf{V}(\mathrm{u}_n)\mathbf{F}(\mathrm{x}_n - \mathrm{x}_{n-1}; \omega)$ (see Eq. (1)).

3.1.1(2) Challenge

This general approach $\mathcal{P}_1$ is computationally intractable, since the likelihood is a non-convex function over a high dimensional space of $[\mathbf{x}; \mathbf{v}]$.

Compressive Sensing-Based Detection

3.1 Problem Formulation

3.1.2(1) Compressive Sensing Perspective

Based on the sparseness structure in Eq. (15), the compressive sensing-based leak detection is formulated as

$$\mathcal{P}_2: \ \hat{\mathbf{u}} = \arg\min_{\mathbf{u}} \|\mathbf{u}\|_0,$$

$$\text{s.t.} \ \|\mathbf{z} - \mathbf{F}(\chi)\mathbf{u}\|_2 \leq \delta, \tag{17}$$

$$\hat{\mathbf{x}} = \{\chi_k | \hat{\mathrm{u}}_k \neq 0, \forall k = 1 : K\}, \tag{18}$$

3.1.2(2) Challenge

- This CS-based ℓ_0-norm optimization $\mathcal{P}_2$ is non-convex.
- The limited representative efficiency of raw base vectors.

Compressive Sensing-Based Detection

3.1 Problem Formulation

3.1.3(1) A relaxed CS formulation

A simultaneous dictionary training and sparse signal identification scheme:

$$\mathcal{P}_3: \ \hat{\chi}, \hat{\mathbf{u}} = \arg\min_{\chi, \mathbf{u}} \|\mathbf{z} - \mathbf{F}(\chi)\mathbf{u}\|_2 + \lambda\|\mathbf{u}\|_1. \tag{19}$$

3.1.3(2) Properties of $\mathcal{P}_3$

- **Merit**
  - The base vectors are optimized with improved representation performance.
  - The algorithmic complexity is insensitive to the number of leaks.
  - The negative effect of non-convex function is mitigated thanks to the sufficient position samples.
- **Defect**
  - The computational overhead is linearly increasing with the number of position samples $\{\chi_k | \forall k = 1 : K\}$.

Compressive Sensing-Based Detection

3.2 The Hierarchical VBI Algorithm

3.2 Solution: The Hierarchical Model-Based Varitional Bayesian Inference

- The hierarchical Bayesian model.
- The variational Bayesian inference algorithm.

Compressive Sensing-Based Detection

3.2 The Hierarchical VBI Algorithm

3.2.1 The Hierarchical Bayesian Model

The objective posteriori distribution is given by ($\alpha = \{\chi, \mathbf{u}, \widetilde{\mathbf{W}}, \mathbf{\Theta}\}$)

$$p(\alpha|\mathbf{z}) \propto \mathcal{N}(\mathbf{z}|\mathbf{F}(\chi)\mathbf{u}, \widetilde{\mathbf{W}})\mathcal{N}(\mathbf{u}|\mathbf{0}, \mathbf{\Theta})\mathcal{W}(\mathbf{\Theta}|\kappa, \mathbf{\Sigma})\mathcal{W}(\widetilde{\mathbf{W}}|\tau, \mathbf{V}). \tag{20}$$

- The sparse nature of CS-based leak detection is retained.
- A theoretically tractable update is preserved at each iteration.

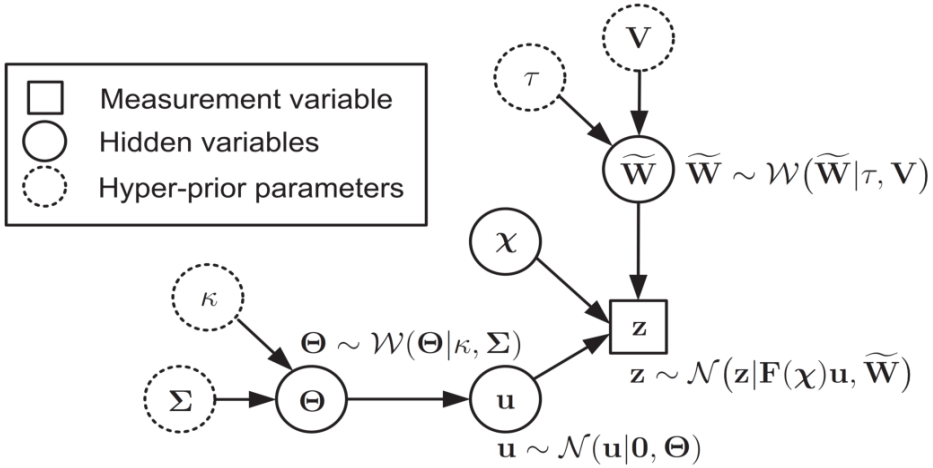


Figure: The hierarchical Bayesian model.

Compressive Sensing-Based Detection

3.2 The Hierarchical VBI Algorithm

3.2.2 The Mean-Field VBI Algorithm

- All unknown variables can be updated distributively, and each individual update only needs local computation (low computational overhead).
- The interaction among those unknown variables can learn further knowledge from each other to reduce system uncertainties (improved detection performance).

The VBI-based joint optimization:,

$$\hat{\boldsymbol{\alpha}} = \arg\min_{q(\boldsymbol{\alpha})} \int q(\boldsymbol{\alpha}) \ln \frac{q(\boldsymbol{\alpha})}{p(\boldsymbol{\alpha}|\mathbf{z})} \mathrm{d}\boldsymbol{\alpha} \tag{21}$$

$$\text{s.t.} \;\; q(\boldsymbol{\alpha}) = \prod_{j:\forall \boldsymbol{\alpha}_j \in \{\boldsymbol{\alpha}\}} q(\boldsymbol{\alpha}_j), \tag{22}$$

$$\int q(\boldsymbol{\alpha}_j)\, \mathrm{d}\boldsymbol{\alpha}_j = 1, \; \forall \boldsymbol{\alpha}_j \in \{\boldsymbol{\alpha}\}, \tag{23}$$

The optimal variational approximation $q(\boldsymbol{\alpha}_j)$, $\forall \boldsymbol{\alpha}_j \in \{\boldsymbol{\alpha}\}$:

$$q(\boldsymbol{\alpha}_j) \propto \exp\left(\left\langle \ln p(\boldsymbol{\alpha}|\mathbf{z}) \right\rangle_{\mathfrak{B}(\boldsymbol{\alpha}_j)} \right), \tag{24}$$

Outline

Numerical Results

A Preliminary Result

A Preliminary Result: CS-Based Detection Without Joint Learning

- For two leaks, $\mathrm{Prob}\left\{ \frac{\|\hat{\mathbf{x}} - \mathbf{x}\|_2}{2} \leq 20[\mathrm{m}] : N = 2 \right\} \geq 0.64$.
- For leaks more than 2, the leak detection performance decays severely.

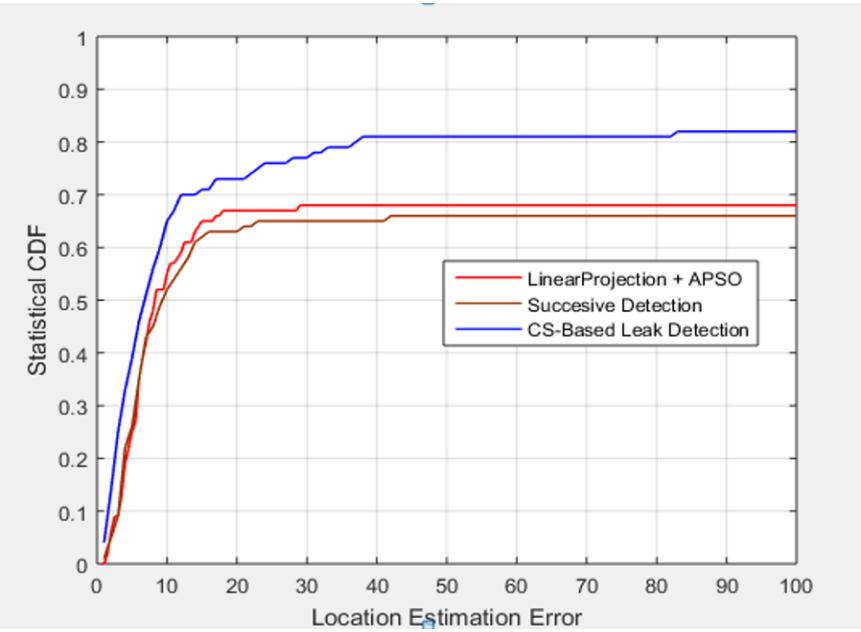


Figure: Multi-leak detection (without joint learning): 2 leaks, 2000[m], 6 sensors, 10 samples.

Numerical Results

A Preliminary Result

A Preliminary Result: CS-Based Detection Without Joint Learning

For 3 leaks, $\mathrm{Prob}\left\{\frac{\|\hat{\mathbf{x}}-\mathbf{x}\|_2}{2} \leq 20[\mathrm{m}] : N=3\right\} \approx 0.3$

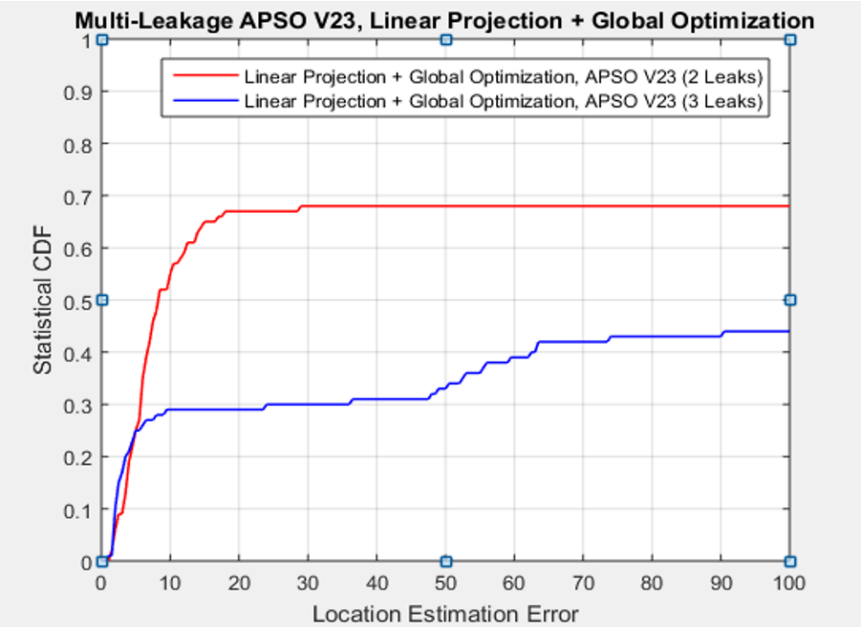


Figure: Multi-leak detection (without joint learning): 3 leaks, 2000[m], 6 sensors, 10 samples.

Outline

Concluding Remarks

Contributions and Insights

Contributions

- A novel compressive sensing-based approach is proposed to solve the complicated detection problem, by exploiting the sparseness structure based on the linear measurement model.
- A hierarchical variational Bayesian inference-based joint learning alg. is proposed to simultaneously optimize the base vectors and estimate the sparse signal.
- The performance limits are analysed to shed lights on the effect of base matrix, position samples, the number of transducers, the deployment of transducers and the number of frequency samples.

Concluding Remarks

Contributions and Insights

Conclusions

- The original multi-leak detection problem is theoretically intractable due to the nonlinear model and the non-convex optimization in high-dimensional space.
- The linear model allows us to exploit a sparseness structure for the multi-leak detection.
- Based on the sparseness structure, a compressive sensing perspective can be employed to solve the complicated leak detection problem.
- In the proposed joint learning scheme, the interaction between base matrix training and leak detection can obtain further information from each other to reduce the system's uncertainties, and thus improving the leak detection performance.

Outline

Future Work

Action Terms

Future Work

- Perform extensive simulations of baseline approaches.
- Complete the simulation of the CS-based leak detection.
- Complete the performance limit analysis.

Thanks for your attention.
Any Question?

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