Linear water hammer equations

Pressure head $H[m]$ and pipe discharge $Q[m^3/s]$ satisfy

$$\begin{aligned} \partial_t H + \frac{a^2}{gA}\partial Q &= 0, \qquad 0 < x < L, \quad t > 0, \\ \partial_t Q + gA\partial H &= 0, \qquad 0 < x < L, \quad t > 0, \\ H = Q &= 0, \qquad 0 < x < L, \quad t = 0 \end{aligned}$$

where

$$\begin{array}{rl} a(x) & \text{wave speed}[m/s] \\ A(x) & \text{pipe cross sectional area}[m^2] \\ g & \text{constant of gravity}[m/s^2] \end{array}$$

Boundary conditions:

unknown at $x = L$

unit impulse discharge $Q(0,t) = \delta_0(t)$ at $x = 0$

Measure: $H(0,t)$. Recover: $a(x)$, $A(x)$?

Solutions to inverse problem

- Gel'fand–Levitan–Krein, Gerver, + many others in the early 70's
- 1D inverse problems, both frequency and time-domain solutions
- Method used here: Sondhi–Gopinath 71
- Modern names and generalized idea: Boundary control method (Belishev 87)
  - control solution on boundary
  - make wave behave as wanted in the interior
  - get information by conservation of mass / momentum

Boundary–interior identity

For simplicity $a(x) = a_0$ constant! Recall

$$\partial_t H + \frac{a^2}{gA} \partial Q = 0$$

and integrate $\int_0^\tau \int_0^{a_0\tau} \dots dxdt$ given any fixed $\tau > 0$.

$$-\int_0^\tau \int_0^{a_0\tau} \partial Q(x,t) dxdt = \int_0^\tau \int_0^{a_0\tau} \frac{gA(x)}{a_0^2} \partial_t H(x,t) dxdt$$

- $H = Q = 0$ at $t = 0$
- hence $H(x,t) = Q(x,t) = 0$ when $x \geqslant a_0 t$, so

$$\int_0^\tau Q(0,t)dt = \int_0^{a_0\tau} \frac{gA(x)}{a_0^2} H(x,\tau)dx \qquad (1)$$

Area recovery

Given solutions Q, H we know from boundary measurements the value of

$$V(\tau) = \int_0^{a_0\tau} \frac{gA(x)}{a_0^2} H(x,\tau) dx.$$

If $Q(0,t) = Q_{1,\tau}(0,t)$ is such that $H = H_{1,\tau}$ and

$$H_{1,\tau}(x,\tau) = \begin{cases} 1, & x < a_0\tau \\ 0, & x \geqslant a_0\tau \end{cases} \quad \text{at } t = \tau,$$

then

$$A(x) = \frac{a_0}{g}(\partial V)\left(\frac{x}{a_0}\right)$$

Calculating a special Q-input

Measurement:

$$Q(0,t) = \delta_0(t)$$
$$H(0,t) = \hat{h}(t)$$

where

$$\hat{h}(t) = \frac{a_0}{gA(0)}(\delta_0(t) + h(t))$$

is the impulse-response function.

If $Q(0,t) = Q_{1,\tau}(0,t)$ at $x = 0$ and

$$Q_{1,\tau}(0,t) + \frac{1}{2}\int_0^{2\tau} Q_{1,\tau}(0,s)h(|s-t|)ds = \frac{gA(0)}{a_0}, \qquad 0 < t < 2\tau$$

then the corresponding pressure wave $H = H_{1,\tau}$ satisfies

$$H_{1,\tau}(x,\tau) = \begin{cases} 1, & x < a_0\tau \\ 0, & x \geqslant a_0\tau \end{cases} \qquad \text{at } t = \tau.$$

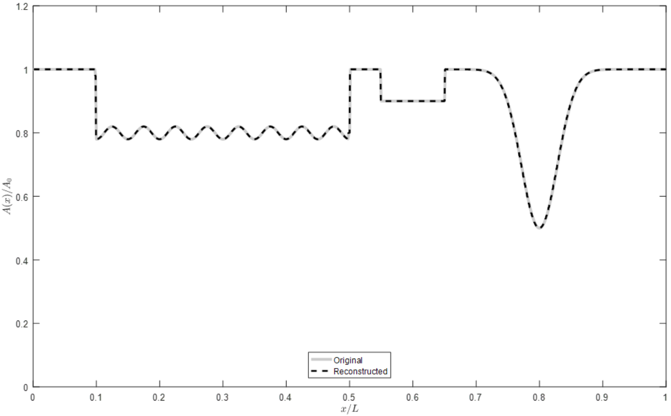
$H_{1,\tau}$ displayed

Algorithm

1. input $Q(0,t) = \delta_0(t)$ and measure $\hat{h}(t) = H(0,t)$ for $t < 2T = 2L/a_0$
2. for $0 < \tau < T$ solve
$$Q_{1,\tau}(0,t) + \frac{1}{2}\int_0^{2\tau} Q_{1,\tau}(0,s)h(|s-t|)ds = \frac{gA(0)}{a_0}, \qquad 0 < t < 2\tau$$
3. set
$$V(\tau) = \int_0^{\tau} Q_{1,\tau}(0,t)dt \qquad \left(= \int_0^{a_0\tau} \frac{gA(x)}{a_0^2}dx\right)$$
4. repeat 2–3 (in the computer) for many τ to get a good approximation of $V(\tau)$
5. given $x < L$ the area can be found by
$$A(x) = \frac{a_0}{g}(\partial V)\left(\frac{x}{a_0}\right).$$

Numerical experiment

Impulse-response simulated using MOC. Area recovered using inversion algorithm.

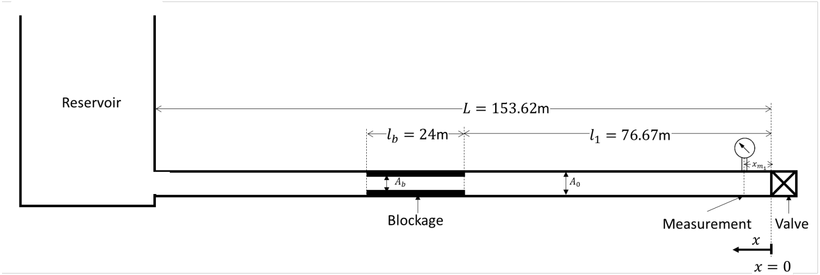


Advantages compared to other methods

- Uses a single measurement location
- Uses a single input discharge wave
- Recovers multiple blockages and their shape (severity)
- Does not require the knowledge of the boundary condition
- Relatively fast and does not involve optimization
- Integral equation, so stable with 0-mean noise

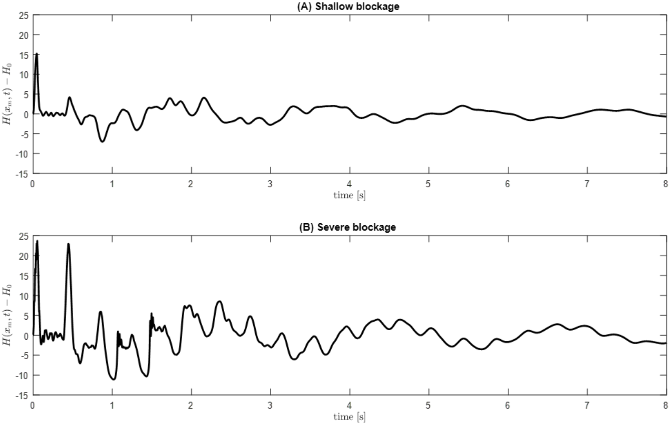
Laboratory experiment: setup

Measurement set up by Silvia Meniconi and Bruno Brunone's group.



Laboratory experiment: impulse-response function

Measurement set up by Silvia Meniconi and Bruno Brunone's group.



Reconstruction from measured and processed data

Measurement set up by Silvia Meniconi and Bruno Brunone's group.