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# STATISTICAL EQUATIONS OF WATER HAMMER

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## 1D – Water-Hammer Equations:

$$\frac{\partial H}{\partial t} + \frac{a^2}{Ag} \left( \frac{\partial Q}{\partial x} \right) = 0$$

$$\frac{\partial Q}{\partial t} + Ag \left( \frac{\partial H}{\partial x} \right) = 0$$

Where,  $H$  = Pressure Head

$Q$  = Flowrate

$a$  = wave-speed

Let  $a^2 = \frac{K}{\rho}$  be random such that:

$$\frac{K}{\rho} = \overline{\left(\frac{K}{\rho}\right)} + \left(\frac{K}{\rho}\right)' \rightarrow a^2 = \overline{a^2} + a^{2'}$$

Therefore,  $H$  and  $Q$  are also random variables:

$$\frac{\partial}{\partial t}(h_o + h_1 + \dots) + \frac{1}{gA} (\overline{a^2} + a^{2'}) \frac{\partial}{\partial x} (q_o + q_1 + \dots) = 0$$

$$\frac{\partial}{\partial t}(q_o + q_1 + \dots) + Ag \frac{\partial}{\partial x}(h_o + h_1 + \dots) = 0$$

Collecting  $0^{\text{th}}$  order terms:

$$\frac{\partial h_o}{\partial t} + \frac{\overline{a^2}}{Ag} \frac{\partial q_o}{\partial x} = 0$$

$$\frac{\partial q_o}{\partial t} + Ag \frac{\partial h_o}{\partial x} = 0$$

1<sup>st</sup> order terms:

$$\frac{\partial h_1}{\partial t} + \frac{1}{gA} \left[ \overline{a^2} \frac{\partial q_1}{\partial x} + a^{2'} \frac{\partial q_o}{\partial x} \right] = 0$$

$$\frac{\partial q_1}{\partial t} + Ag \frac{\partial}{\partial x} \frac{\partial h_1}{\partial t} = 0$$

Cross correlated 1<sup>st</sup> order terms with  $h$ , gives:

$$\frac{\partial}{\partial t} \langle h_1, h_1 \rangle + \frac{\overline{a^2}}{Ag} \frac{\partial}{\partial x} \langle q_1, h_1 \rangle + \frac{1}{Ag} \frac{\partial q_o}{\partial x} (\langle a^{2'}, h_1 \rangle) = 0$$

$$\frac{\partial}{\partial t} \langle q_1, h_1 \rangle + Ag \frac{\partial}{\partial x} \langle h_1, h_1 \rangle = 0$$

Cross correlated (1) and (2) with  $q_1$ :

$$\frac{\partial}{\partial t} \langle h_1, q_1 \rangle + \frac{\overline{a^2}}{Ag} \frac{\partial}{\partial x} \langle q_1, q_1 \rangle + \frac{1}{Ag} \frac{\partial q_o}{\partial x} (\langle a^{2'}, q_1 \rangle) = 0$$

$$\frac{\partial}{\partial t} \langle q_1, q_1 \rangle + Ag \frac{\partial}{\partial x} \langle h_1, q_1 \rangle = 0$$

Cross correlate (1) and (2) with  $a^{2'}$ :

$$\frac{\partial}{\partial t} \langle h_1, a^{2'} \rangle + \frac{\overline{a^2}}{Ag} \frac{\partial}{\partial x} \langle q_1, a^{2'} \rangle + \frac{1}{Ag} \frac{\partial q_o}{\partial x} (\langle a^{2'}, a^{2'} \rangle) = 0$$

$$\frac{\partial}{\partial t} \langle q_1, a^{2'} \rangle + Ag \frac{\partial}{\partial x} \langle h_1, a^{2'} \rangle = 0$$

- 0<sup>th</sup> Order Terms:

$$\frac{\partial h_o}{\partial t} + \frac{\bar{a^2}}{Ag} \frac{\partial q_o}{\partial x} = 0$$

• || Equation 01

$$\frac{\partial q_o}{\partial t} + Ag \frac{\partial h_o}{\partial x} = 0$$

• || Equation 02

- 1<sup>st</sup> Order Terms:

Cross correlation with  $h_1$ :

$$\frac{\partial}{\partial t} \sigma_{h_1}^2 + \frac{\bar{a^2}}{Ag} \frac{\partial}{\partial x} cov(h_1, q_1) + \frac{1}{Ag} cov(a^{2'}, h_1) \frac{\partial q_o}{\partial x} = 0$$

• || Equation 03

$$\frac{\partial}{\partial t} cov(q_1, h_1) + Ag \frac{\partial \sigma_{h_1}^2}{\partial x} = 0$$

• || Equation 04

Cross Correlation with  $q_1$ :

$$\frac{\partial}{\partial t} cov(h_1, q_1) + \frac{\bar{a^2}}{Ag} \frac{\partial \sigma_{q_1}^2}{\partial x} + \frac{1}{Ag} cov(a^{2'}, q_1) \frac{\partial q_o}{\partial x} = 0$$

• || Equation 05

$$\frac{\partial \sigma_{q_1}^2}{\partial t} + Ag \frac{\partial cov(h_1, q_1)}{\partial x} = 0$$

• || Equation 06

Cross Correlation with  $a^{2'}$ :

$$\frac{\partial}{\partial t} cov(a^{2'}, h_1) + \frac{\bar{a^2}}{Ag} \frac{\partial}{\partial x} (cov(a^{2'}, q_1)) + \frac{1}{Ag} (\sigma_{a^{2'}}^2) \frac{\partial q_o}{\partial x} = 0$$

• || Equation 07

$$\frac{\partial}{\partial t} cov(a^{2'}, q_1) + Ag \frac{\partial}{\partial x} (cov(a^{2'}, h_1)) = 0$$

• || Equation 08  
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- Taking Equation 7 & 8:

$$\frac{\partial}{\partial t} cov(a^{2'}, h_1) + \frac{\overline{a^2}}{Ag} \frac{\partial}{\partial x} (cov(a^{2'}, q_1)) + \frac{1}{Ag} (\sigma_{a^{2'}}^2) \frac{\partial q_o}{\partial x} = 0$$

$$\frac{\partial}{\partial t} cov(a^{2'}, q_1) + Ag \frac{\partial}{\partial x} (cov(a^{2'}, h_1)) = 0$$

- Assumptions:

$\sigma_{a^{2'}}^2 \rightarrow$  Known

$\frac{\partial q_o}{\partial x} \rightarrow$  estimated using 1<sup>st</sup> order approximation

$$[cov(a^{2'}, q_1)]_P = \frac{(C_{P1} + C_{N1})}{2}$$

$$[cov(a^{2'}, h_1)]_P = \frac{(C_{P1} - C_{N1})}{2C_a}$$

Where,

$$C_a = \frac{Ag}{\bar{a}}$$

$$C_{P1} = [cov(a^{2'}, q_1)]_A + C_a [cov(a^{2'}, h_1)]_A - \frac{1}{\bar{a}} (\sigma_{a^{2'}}^2) \frac{\partial q_o^A}{\partial x} \Delta t$$

$$C_{N1} = [cov(a^{2'}, q_1)]_B - C_a [cov(a^{2'}, h_1)]_B + \frac{1}{\bar{a}} (\sigma_{a^{2'}}^2) \frac{\partial q_o^A}{\partial x} \Delta t$$

$$[cov (q_1, h_1)]_P = \frac{(C_{P2} + C_{N2})}{2}$$

$$[\sigma_{H1}^2]_P = \frac{(C_{P2} - C_{N2})}{2 C_a}$$

Where,

$$C_{P2} = [cov (q_1, H_1)]_A + C_a \left[ (\sigma_{H1}^2)_A \right] - \frac{1}{\bar{a}} \left[ cov (a^{2'}, H_1) \right]^A \frac{\partial Q_o^A}{\partial x} \Delta t$$

$$C_{N2} = [cov (Q_1, H_1)]_B - C_a \left[ (\sigma_{H1}^2)_B \right] + \frac{1}{\bar{a}} \left[ cov (a^{2'}, H_1) \right]^A \frac{\partial Q_o^A}{\partial x} \Delta t$$

$$[cov(a^{2'}, Q_1)]_P = \frac{(\mathcal{C}_{P1} + \mathcal{C}_{N1})}{2}$$

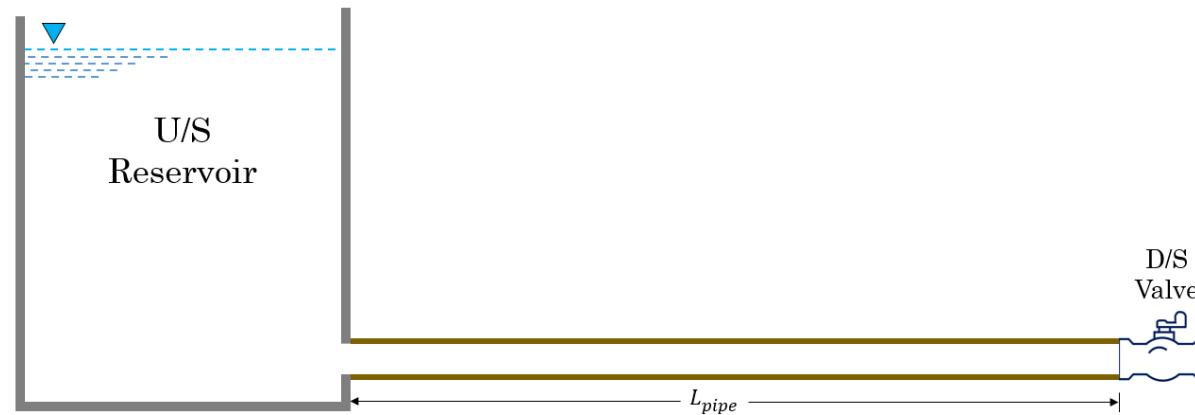
$$[cov(a^{2'}, H_1)]_P = \frac{(\mathcal{C}_{P1} - \mathcal{C}_{N1})}{2\mathcal{C}_a}$$

Where:

$$\mathcal{C}_{P3} = [\sigma_{Q_1}^2]_A + \mathcal{C}_a [cov(H_1, Q_1)]_A - \frac{1}{\bar{a}} [cov(a^{2'}, Q_1)]^A \frac{\partial Q_o^A}{\partial x} \Delta t$$

$$\mathcal{C}_{N3} = [\sigma_{Q_1}^2]_B - \mathcal{C}_a [cov(H_1, Q_1)]_B + \frac{1}{\bar{a}} [cov(a^{2'}, Q_1)]^A \frac{\partial Q_o^A}{\partial x} \Delta t$$

## CASE: Reservoir Pipe Valve



### U/S Boundary Conditions:

- $Var(h_1) = 0$
- $Var(q_1) = \text{Neg. Charc.}$
- $Var(a^{2'}) = 15\% \text{ of } \overline{a^2}$
- $cov(h_1, q_1) = 0$
- $cov(h_1, a^{2'}) = 0$
- $cov(q_1, a^{2'}) = 0$

### Initial Conditions:

- $Var(h_1) = 0$
- $Var(q_1) = 0$
- $Var(a^{2'}) = 15\% \text{ of } \overline{a^2}$
- $cov(h_1, q_1) = 0$
- $cov(h_1, a^{2'}) = 0$
- $cov(q_1, a^{2'}) = 0$

### D/S Boundary Conditions:

- $Var(h_1) = \text{Pos. Charc.}$
- $Var(q_1) = 0$
- $Var(a^{2'}) = 0$
- $cov(h_1, q_1) = 0$
- $cov(h_1, a^{2'}) = 0$
- $cov(q_1, a^{2'}) = 0$

# ANALYTICAL SOLUTION OF WATER-HAMMER EQUATIONS

The one dimensional water hammer equations are given below:

$$a^2 \frac{\partial Q}{\partial x} + gA \frac{\partial H}{\partial t} = 0$$
$$\frac{\partial Q}{\partial t} + gA \frac{\partial H}{\partial x} = 0$$

$Q$  and  $H$  is divided into two parts: “Steady” & “Unsteady”:

$$Q = Q_s + q$$
$$H = H_s + h$$

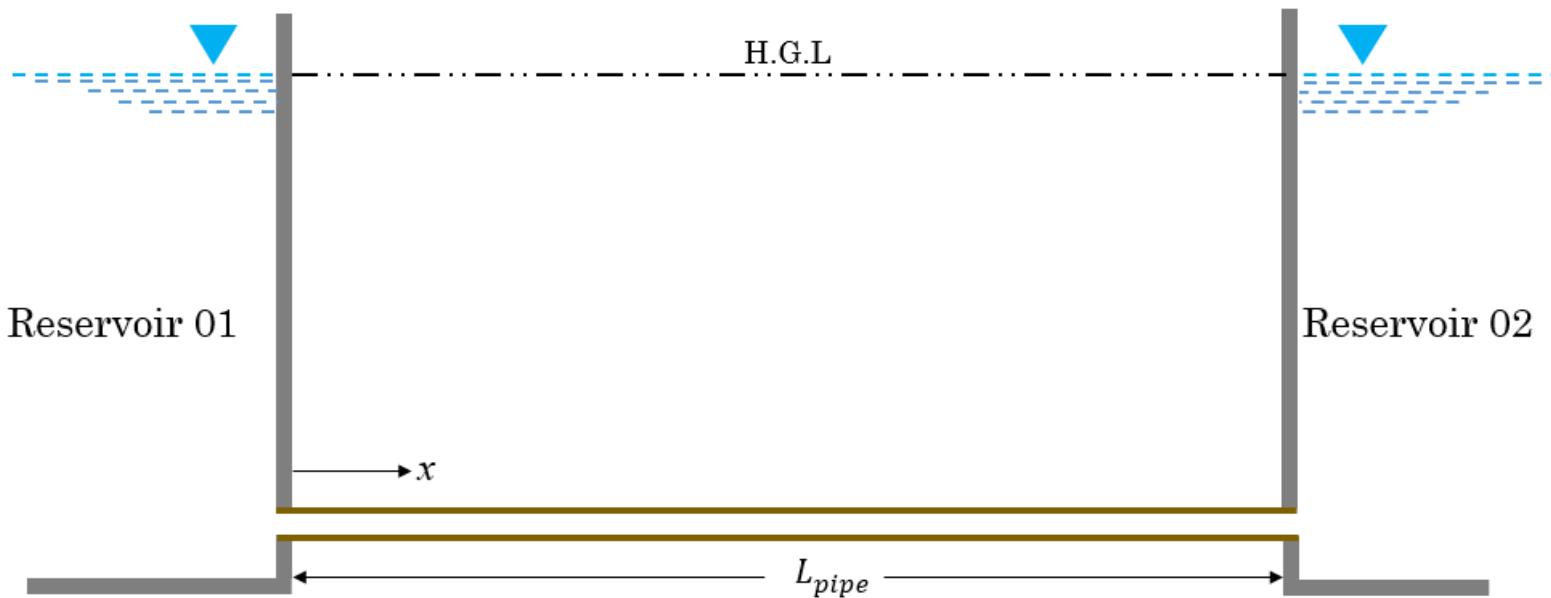
Substitute  $Q$  and  $H$  in continuity and momentum equation:

$$a^2 \frac{\partial q}{\partial x} + gA \frac{\partial h}{\partial t} = 0$$
$$\frac{\partial q}{\partial t} + gA \frac{\partial h}{\partial x} = 0$$

The governing equation for wave is:

$$\frac{\partial^2 h}{\partial t^2} - a^2 \frac{\partial^2 h}{\partial x^2} = 0$$

## CASE: Reservoir Pipe Reservoir



Boundary condition for upstream & downstream reservoir :

$$h(x_0, t) = 0$$

$$h(x_L, t) = 0$$

The harmonic solution for  $h$  is:

$$h(x, t) = \sum_n A_n(t) \sin\left(\frac{n\pi}{L}\right) x$$

$$\frac{\partial^2 h}{\partial t^2} - a^2 \frac{\partial^2 h}{\partial x^2} = 0$$

$$\sum_n \sin\left(\frac{n\pi}{L}\right) x \left[ A''_n(t) + a^2 A_n(t) \frac{n^2 \pi^2}{L^2} \right] = 0 \quad (1)$$

Taking the inner product of the equation (1) gives,

$$\sum_n \int_0^L \sin\left(\frac{n\pi}{L}\right) x \cdot \sin\frac{m\pi x}{L} dx \left[ A''_n(t) + a^2 A_n(t) \frac{n^2 \pi^2}{L^2} \right] = 0$$

$$\text{for } n \neq m, \int_0^L \sin \frac{n\pi x}{L} \cdot \sin \frac{m\pi x}{L} dx = 0$$

$$\text{for } n = m ; \quad \int_0^L \sin \frac{n\pi x}{L} \cdot \sin \frac{m\pi x}{L} dx \neq 0$$

Thus,

$$\sum_n \int_0^L \sin \left( \frac{n\pi}{L} \right) x \cdot \sin \frac{m\pi x}{L} dx \left[ A''_n(t) + a^2 A_n(t) \frac{n^2 \pi^2}{L^2} \right] = 0$$

$$A''_m(t) + a^2 A_m(t) \frac{m^2 \pi^2}{L^2} = 0$$

Assume  $A_m(t) \cong e^{i\lambda t}$

$$\begin{aligned} -\lambda^2 e^{i\lambda t} + \frac{a^2 m^2 \pi^2}{L^2} e^{i\lambda t} &= 0 \\ e^{i\lambda t} \left( -\lambda^2 + \frac{a^2 m^2 \pi^2}{L^2} \right) &= 0 \end{aligned}$$

Here,  $e^{i\lambda t} \neq 0$

$$-\lambda^2 + \frac{a^2 m^2 \pi^2}{L^2} = 0$$

$$\lambda = \pm \frac{am\pi}{L} = \pm \omega$$

The solution for  $A_m(t)$  is

$$A_m(t) = C_1 e^{i\omega t} + C_2 e^{-i\omega t}$$

The general form for  $h$  is:

$$h_m(x, t) = (C_1 e^{i\omega t} + C_2 e^{-i\omega t}) \sin\left(\frac{m\pi}{L}\right) x$$

Using continuity equation we get,

$$a^2 \frac{\partial q(x, t)}{\partial x} = -gA \frac{\partial h(x, t)}{\partial t}$$

$$a^2 \frac{\partial q(x, t)}{\partial x} = -\frac{gA}{a^2} (i\omega C_1 e^{i\omega t} - i\omega C_2 e^{-i\omega t}) \sin\left(\frac{m\pi}{L}\right) x \quad (2)$$

Integrate equation (2) with respect to  $\xi$ ,

$$\int_0^x \frac{\partial q(\xi, t)}{\partial \xi} d\xi = - \int \frac{gA}{a^2} (i\omega C_1 e^{i\omega t} - i\omega C_2 e^{-i\omega t}) \sin\left(\frac{m\pi}{L}\right) \xi d\xi$$

The general form for  $q$  is:

$$q_m(x, t) = \frac{gAi}{a} (C_1 e^{i\omega t} - C_2 e^{-i\omega t}) \left[ \cos\left(\frac{m\pi}{L}\right) x - 1 \right]$$