Sampling-type Reconstruction Methods for Inverse Scattering Problems

Jun Zou
Department of Mathematics
The Chinese University of Hong Kong

Joint work with

Bangti Jin (UC Riverside) & Jingzhi Li (SUSTC)
Hongyu Liu (UNCC) & Keji Liu (SUFE)

http://www.math.cuhk.edu.hk/~zou

OUTLINE

- Inverse Obstacle/Medium Scattering Problems

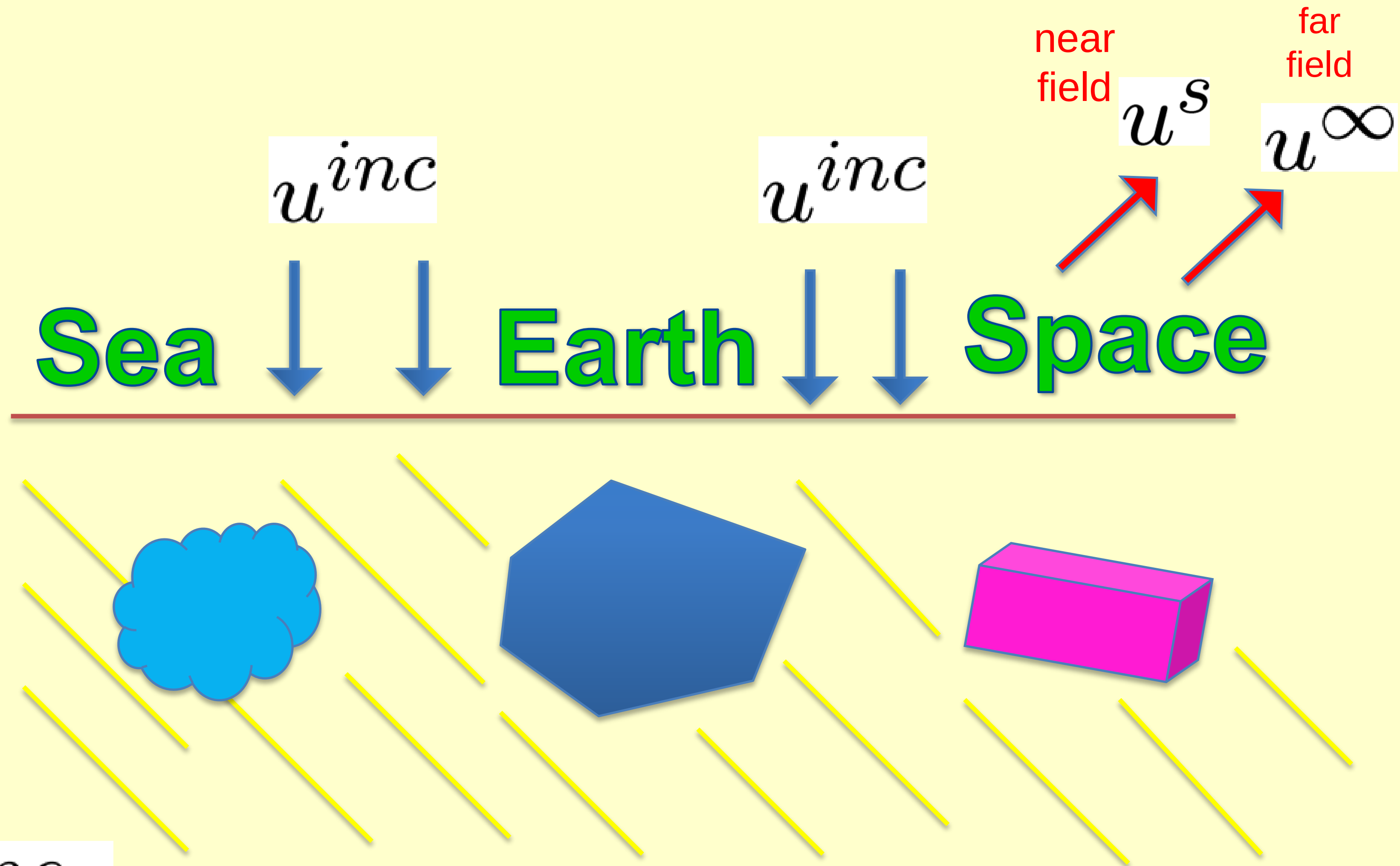
  & Mathematical Understandings

- *Sampling-type Methods*

- Direct Sampling Methods

- Two-stage semi-smooth Newton Method

Inverse Media/Obstacle Scattering Problem



u^{inc} : **Acoustic, electromagnetic, elastic waves**

Acoustic Obstacle/Media Scattering

- Take the planar incident field

$$u^i = \exp(ikx \cdot d),$$

- then the total field $u = u^i + u^s$ solves

$$\Delta u + k^2 u = 0 \quad \text{in} \quad G = \mathbb{R}^N \setminus D,$$

$$\Delta u + k^2 n^2(x) u = 0 \quad \text{in} \quad \mathbf{R}^n$$

- u^s satisfies the Sommerfeld radiation condition:

$$\lim_{r\to\infty} r^{(N-1)/2}\left(\frac{\partial u^s}{\partial r} - iku^s\right) = 0$$

Reflective Index

- Reflective index :

$$n(x) = \frac{\text{sound speed in homogeneous medium}}{\text{sound speed at position } x}$$

- Reflective index is known, then medium is known：

air, wooden, metal, human, … …

Physical Properties of Obstacles

- Physical Properties of obstacles:

Sound-soft: $u = 0 \text{ on } \partial D$ (pressure vanishes)

Sound-hard: $\partial u / \partial \nu = 0 \text{ on } \partial D$
(normal velocity of wave vanishes)

Impedance: $\partial u / \partial \nu + i\,\lambda(x)\, u = 0 \text{ on } \partial D$
(normal velocity proport. to pressure)

or mixed type

Identifiability of Acoustic Obstacles

- **A long-standing problem :**

  Far field data from how many incident fields

  is sufficient to uniquely determine a scatterer, consisting of

  many separated objects of different physical properties ?

- **Numerous results, answers quite limited till 10 years ago:**

  for general obstacles with known physical properties ,

  needs far field data from countably infinitely many incident fields

Recent Advances on Identifiability

- Cheng, Yamamoto, Elschner 03, 06 : reflection principle
  A single sound-hard polygonal obstacle, by at most 2 incidents

- Restricted D to the class of polyhedral types

- Liu-Zou 06, 07, 08 : by N incident fields

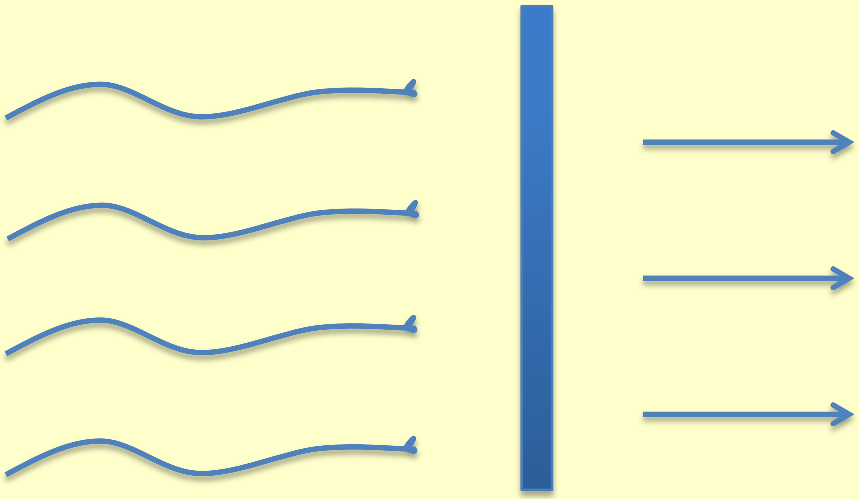
  D consists of finitely many polyhedral obstacles,
  either sound-soft, sound-hard, or of mixed type
  or contain sound-soft crack-type components

Two Key Mathematical Tools

- **Basic tools for acoustic waves:**

Reflection principle & Path argument

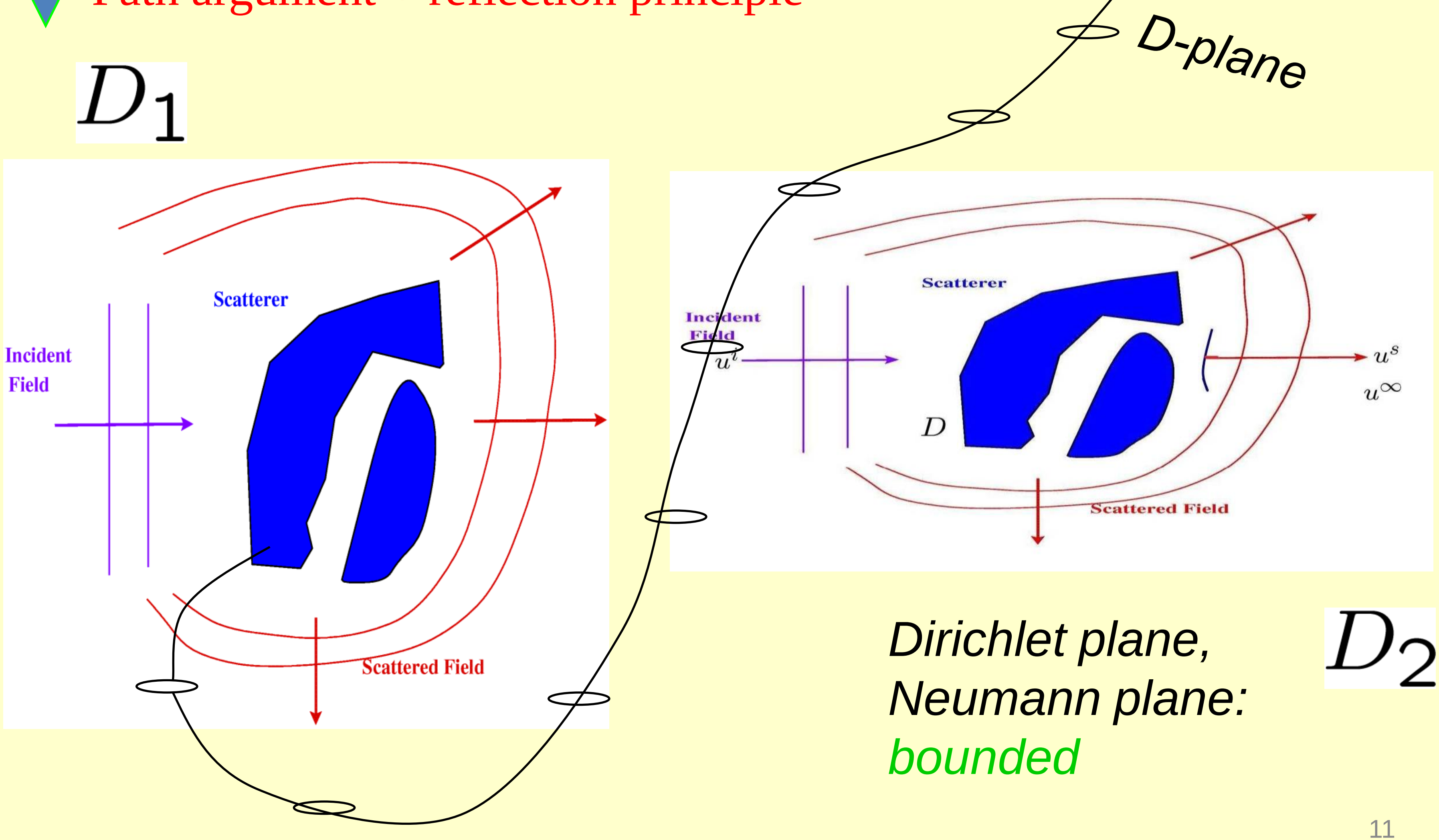
Reflection Principle of Acoustic Wave



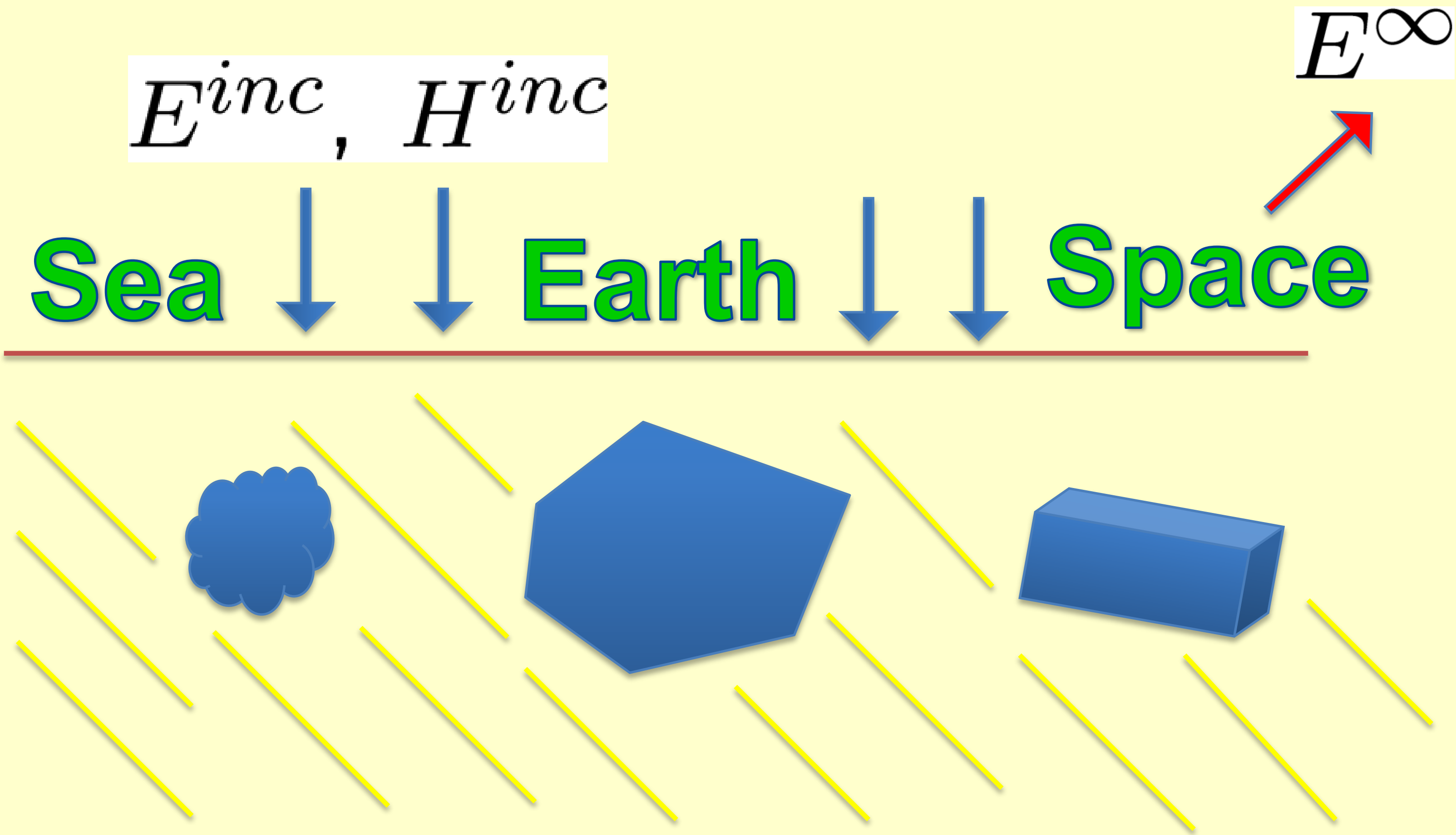
sound wave

Basic Ideas in Proofs

- Path argument + reflection principle



Identifiability: Inverse EM Obstacle Scattering



Two Key Ingredients to EM Waves
(Liu-Yamamoto-Zou 07)

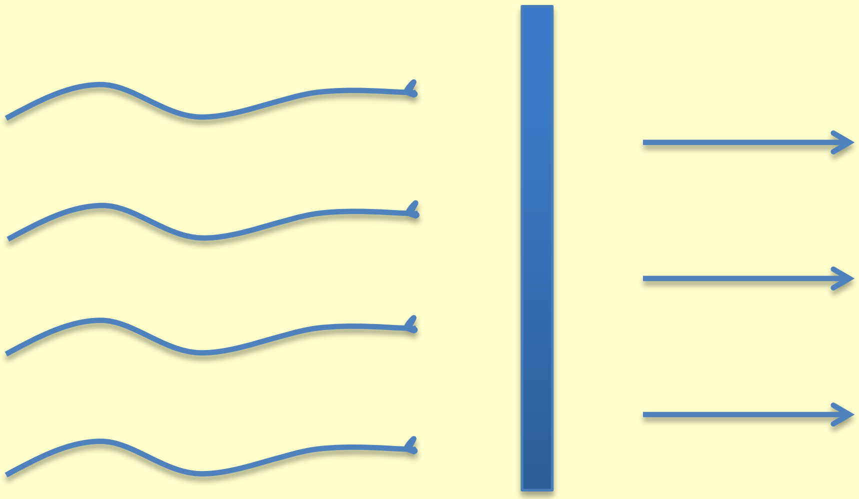
- **Basic tools for acoustic waves:**

  Path argument & reflection principle

- Path argument works for all wave models :

  sound waves, EM waves, elastic waves

Reflection Principle of EM Wave



Electromagnetic wave

Reflection Principle For Maxwell Equations
(Liu-Yamamoto-Zou 07)

- **Any hyperplane :** Π

$G = \mathbb{R}^N \setminus D$

If $\nu \times \mathbf{E} = 0$ on Π (resp. $\nu \times (\mathrm{curl} \times \mathbf{E}) = 0$),

Then the following BCs can be reflected w.r.t. any hyperplane Π in G:

$$\nu \times \mathbf{E} = 0$$

$$\nu \times (\mathrm{curl} \times \mathbf{E}) = 0$$

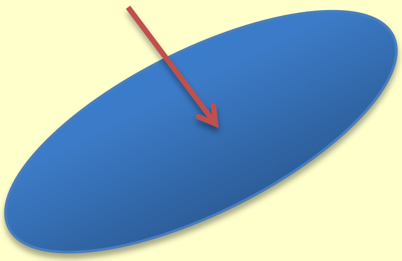
$$\nu \times \mathrm{curl}\, \mathbf{E} - \mathrm{i}\, \lambda\, (\nu \times \mathbf{E}) \times \nu = 0$$

Inverse EM Obstacle Scattering

- **Liu-Yamamoto-Zou 08 & Liu-Zhang-Zou 09 :**

  Far field data from *1 incident EM* field :
  sufficient to determine
  general polyhedral type obstacles

$$\mathbf{E}^{inc} = \mathbf{s}\, e^{(i\mathbf{q}\cdot\mathbf{x})}$$



- Bao-Zhang-Zou (Trans. AMS 11; 12) :

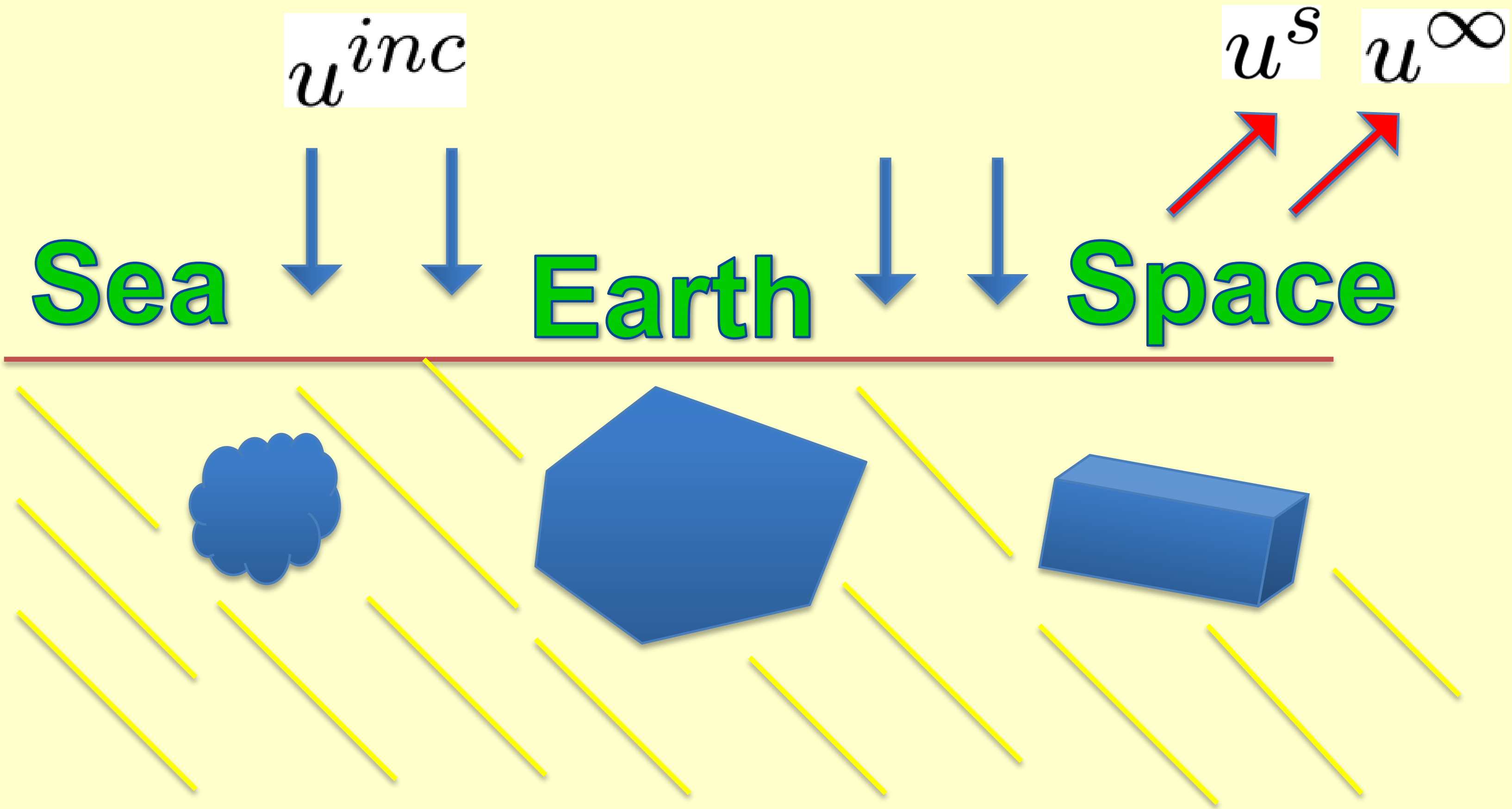
  periodic D : micro-optics, grating structure ;

  Apply EM reflection principle & dihedral group theory

  To classify all unidentifiable gratings into 3 groups

How To Reconstruct :

Shape, Location, Physics ?



Existing Numerical Methods

- **Colton, Kress et al.:**

  Newton-type methods - highly nonlinear eqns
- **Gruber et al.:**

  Multiple Signal Classification (MUSIC) - small scatters
- **Colton, Kirsch et al.:**

  Linear Sampling-type Methods - blowing up indicators
- **van den Berg et al.:**

  Contrast Source Inversion - nonlinear iterative optimiz
- **XD Chen et al.:**

  Subspace-based optimizations – CSVD + CSI

… …

Newton-type Algorithms for Reconstructions

- Most methods target at solving the nonlinear eqn

$$F(\partial D) = u_\infty(\widehat{x}; d)$$

- Highly nonlinear, severely ill-posed;

Popular: numerous variants of Newton's method,

need good initial guess & physical properties of D,
repeated forward solutions,
need the derivatives of u^s w.r.t. changes of ∂D,
characterize evolving of the approximate boundary

Linear Sampling Method

- Colton-Kirsch 96 : a very simple idea, motivated by elegant math observations

- Consider the far-field operator $F : L^2(S^{N-1}) \mapsto L^2(S^{N-1})$

$$(Fg)(\hat{x}) = \int u_\infty(\hat{x}, d)\, g(d)\, ds(d)\,, \quad \hat{x} \in S^{N-1}$$

and the far-field equation for g:

$$Fg = \Phi_\infty(\cdot, z)$$

$$\Phi_\infty(\hat{x}, z) = \gamma e^{-ik\hat{x}\cdot z},$$

$$\forall\, z \in R^N$$

- Clearly $g = g(\hat{x}, z)$. Look at the energy of g :

$$\|g(\cdot, z)\|_{L^2(S^{N-1})}$$

Algorithm of LSM

- Turns reconstructing D into computing indicator $g(\cdot, z)$

- Algorithm of LSM : Select a cut-off value c

1. Select a grid T_h of sampling points, covering D
2. At each z , solve the far-field equation for $g(\cdot, z)$
3. Determine

$$z \in D \text{ if } \|g(\cdot, z)\| \leq c \text{;}$$

$$z \notin D \text{ if } \|g(\cdot, z)\| > c$$

Advantages of LSM

- No need to know the physical properties of ***D*** ;

  No need to approximate geometrical boundaries of domains ;

  No iterations & optimizations

Drawbacks of LSM

- No effective strategies to choose cut-off values.

- Huge computational efforts:
  need to solve the far-field equation for each sampling point, e.g.,

  for an $n \times n \times n$ grid, need to solve $\boldsymbol{n^3}$ ill-posed equations

  The grid should be very fine to get a fine reconstruction

New Techniques

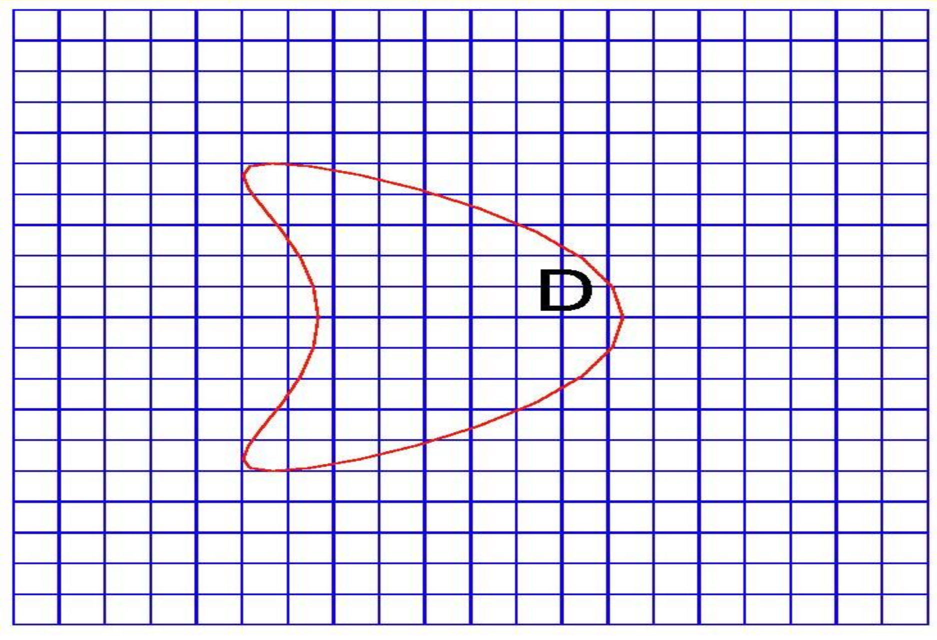
- Li-Liu-Zou, SISC 09:
  Multilevel Linear Sampling Method,
  reduce computational complexity from $O(n^3)$ to $O(n^2)$

- Li-Liu-Zou, SISC 10:
  Strengthened LSM with a Reference Obstacle,
  provide a deterministic technique to select cut-off values

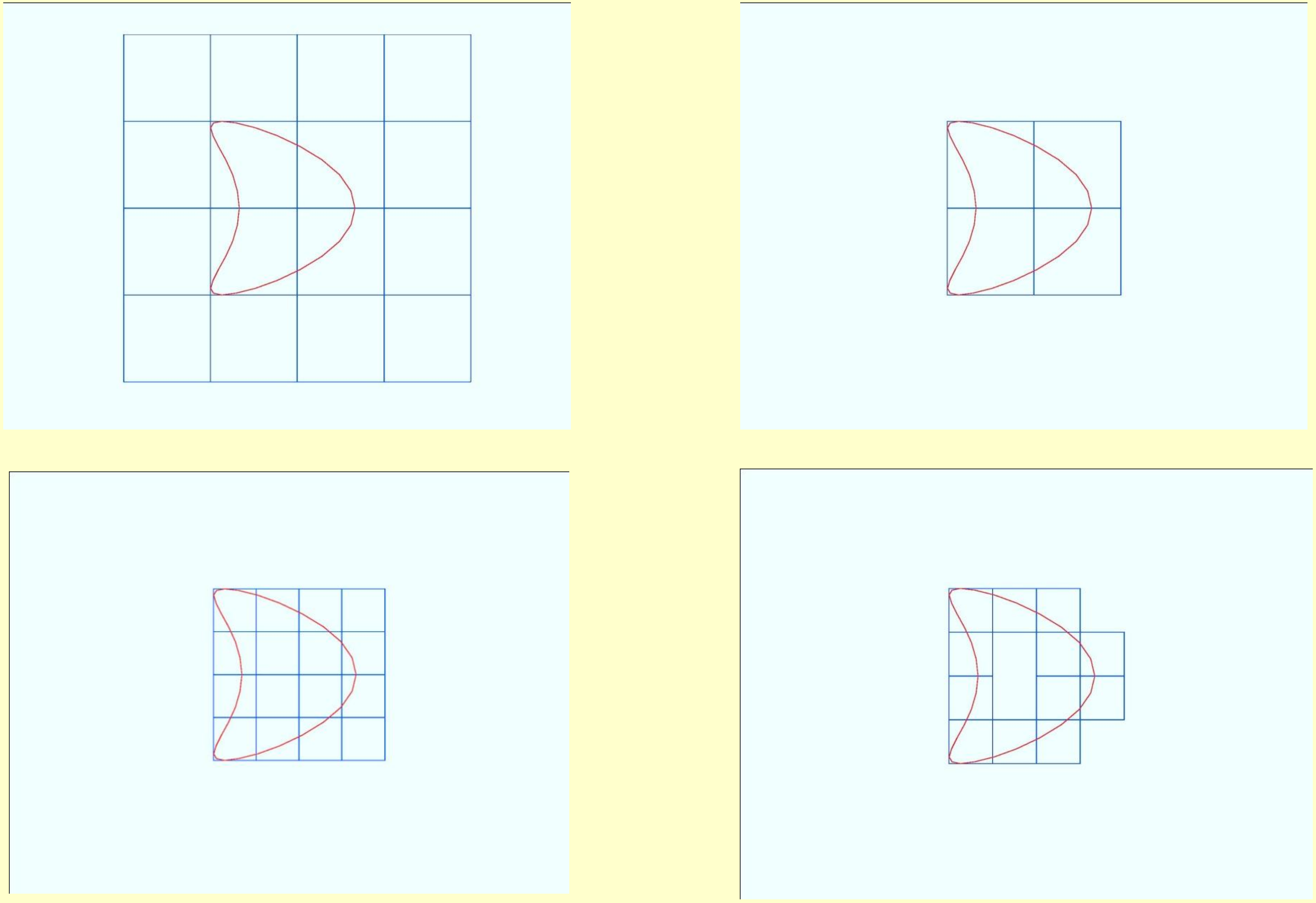
- Liu-Zou, Inv Prb Sci Eng 12:
  Radial Bisection Algorithm, with complexity $O(\log_2 n)$

Multilevel Linear Sampling Method

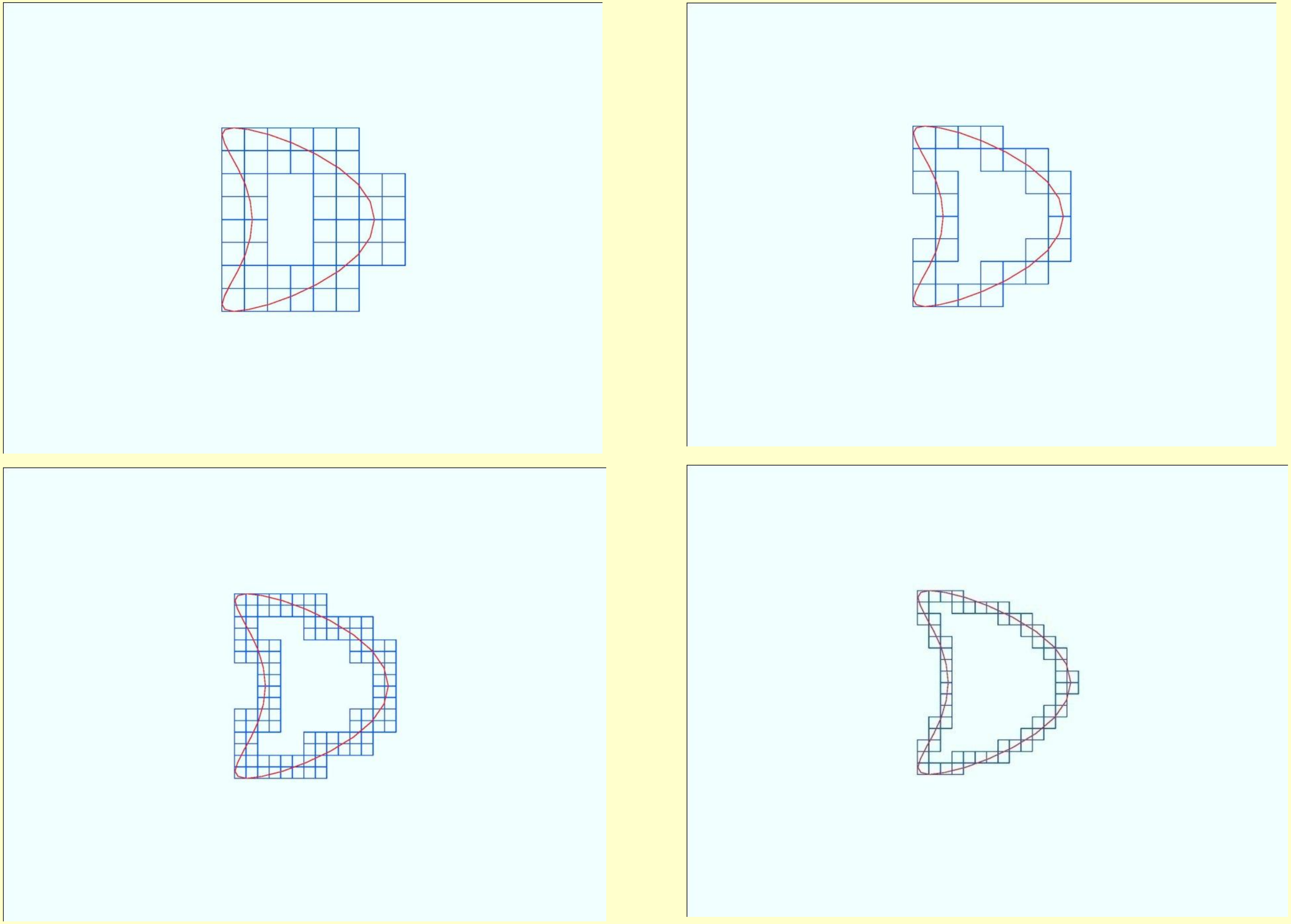
- MLSM : get rid of remote and inner cells



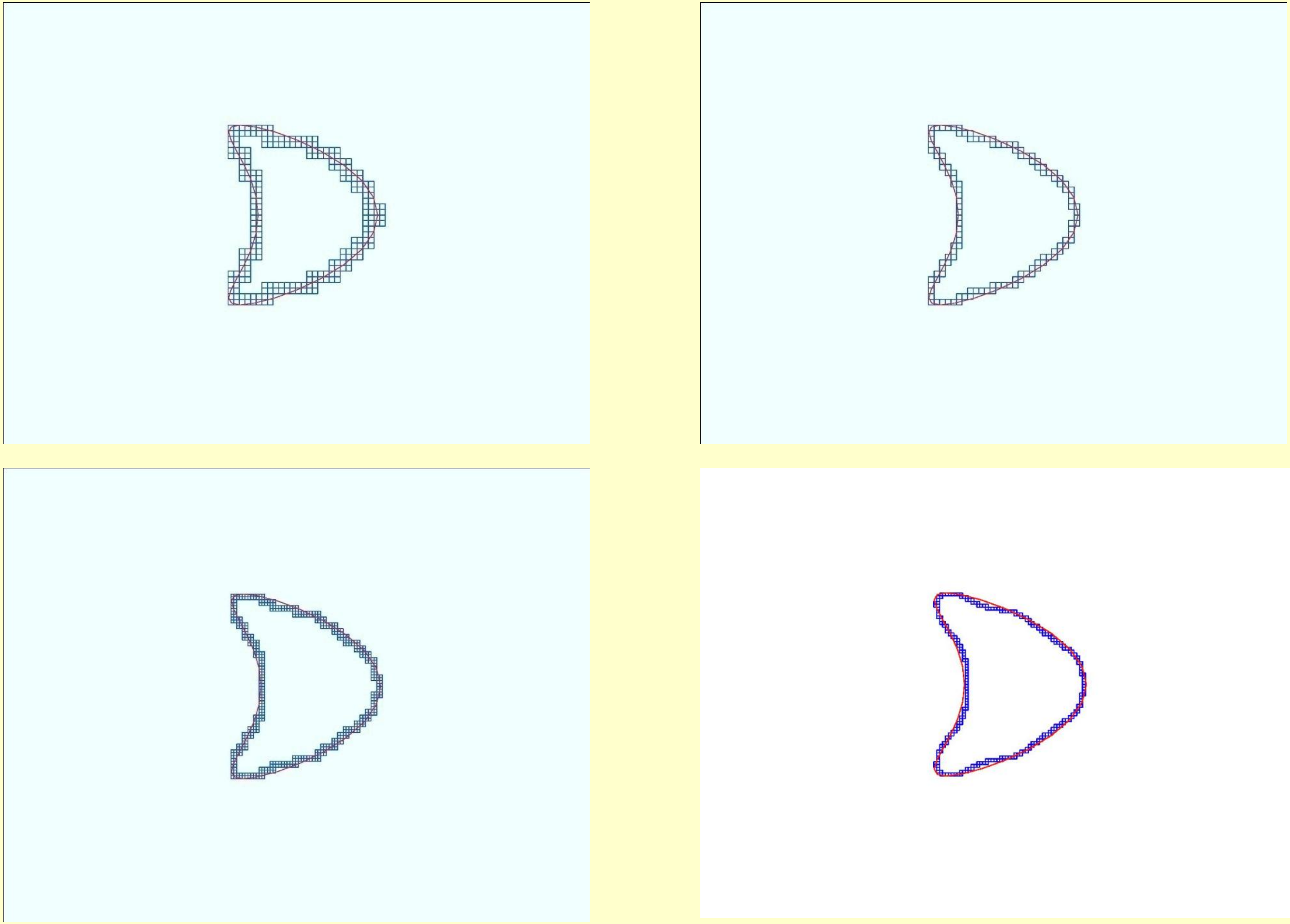
Numerical Example I



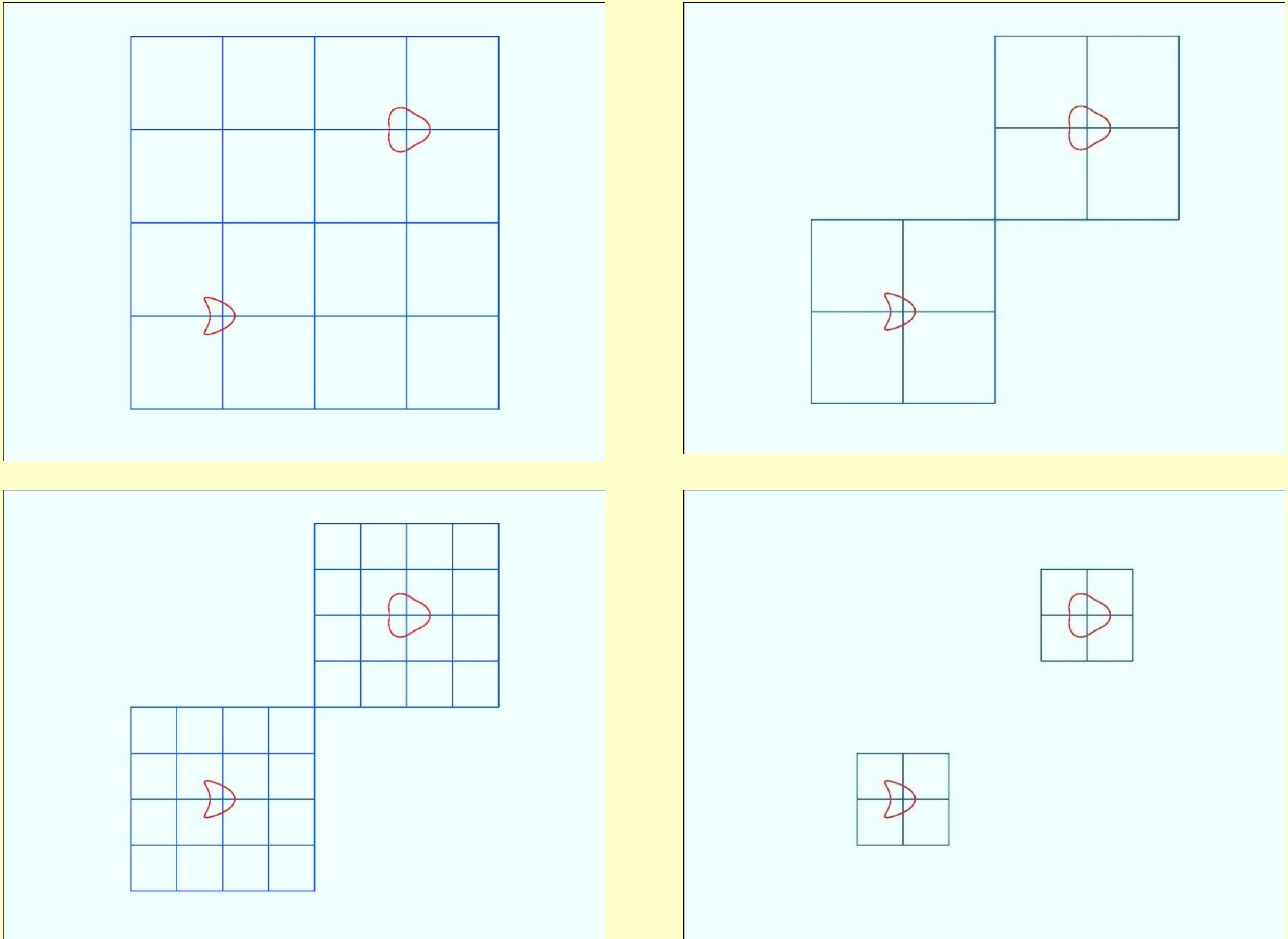
Numerical Example I



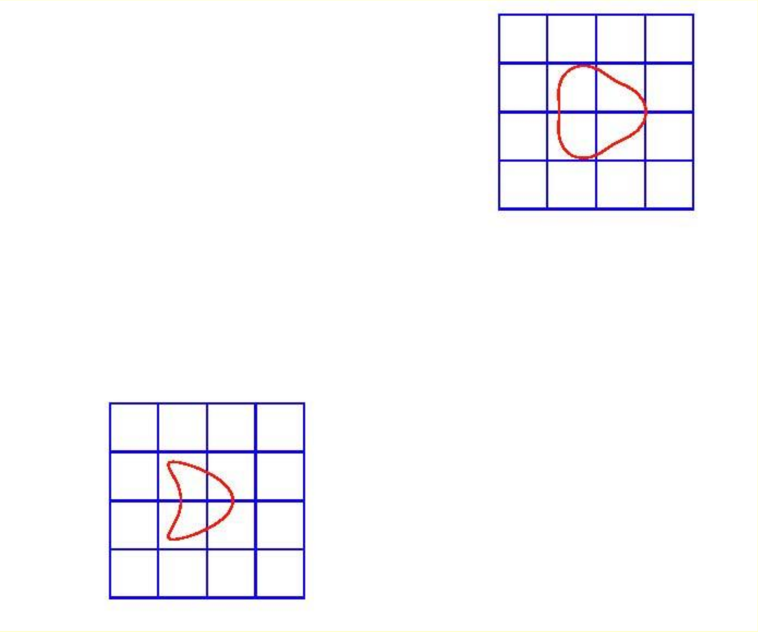
Numerical Example I

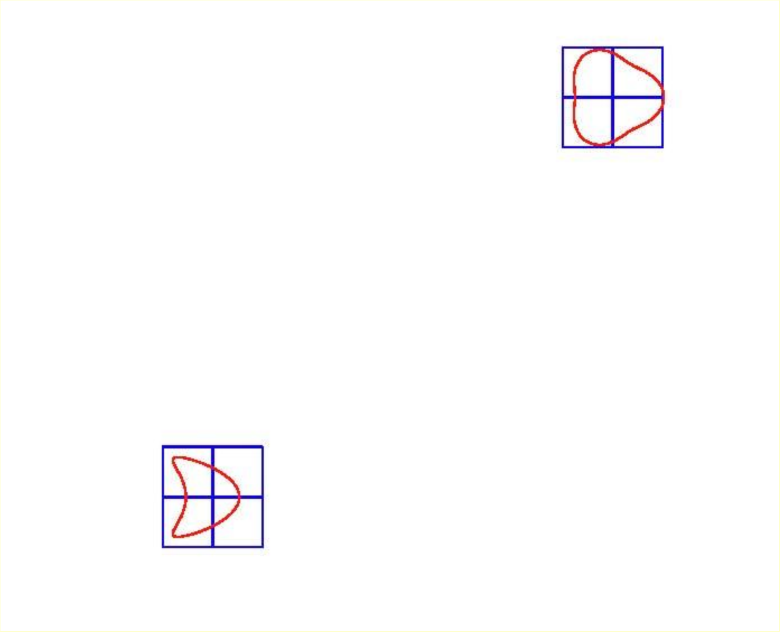


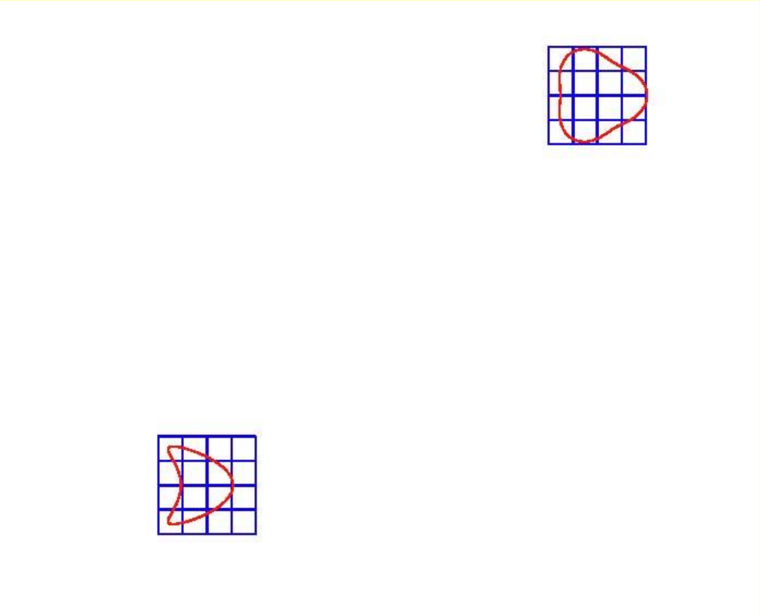
Numerical Example II

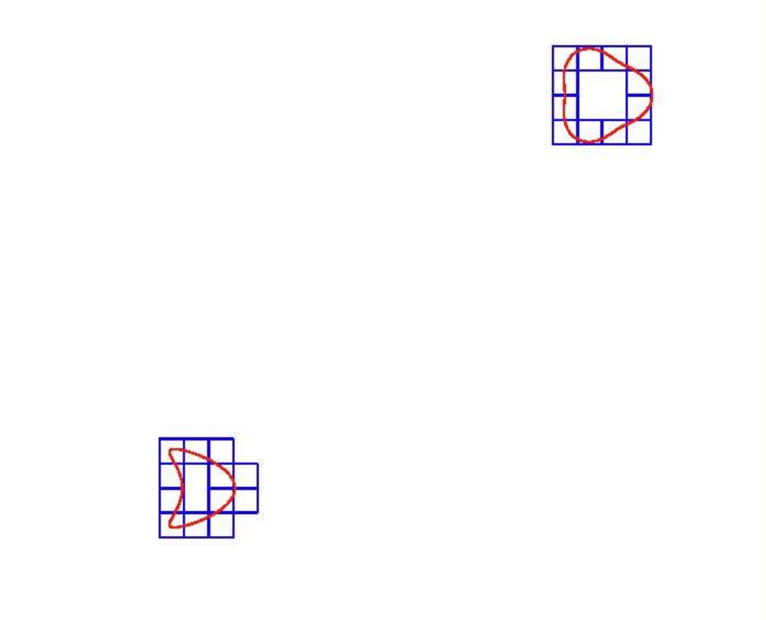


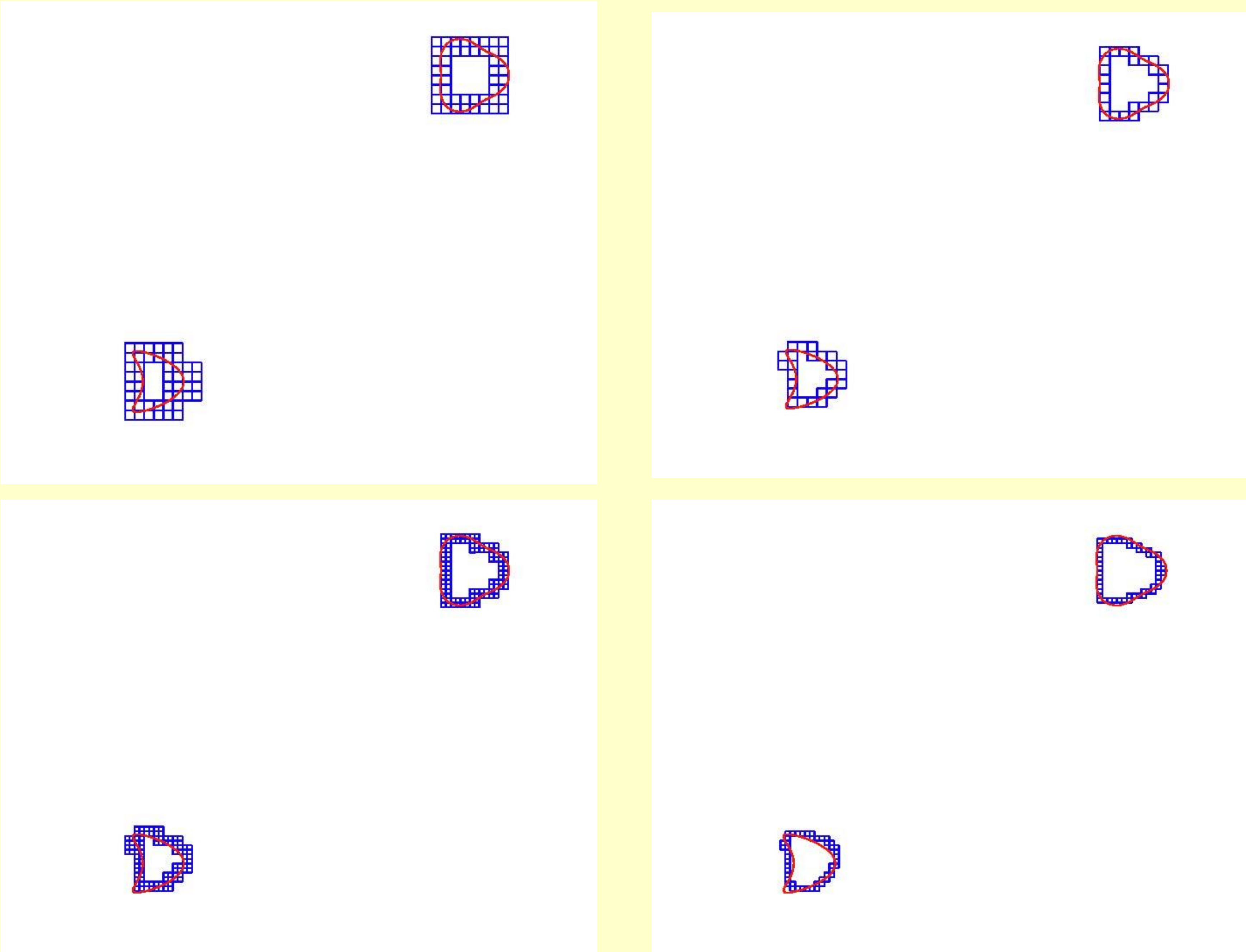
Numerical Example II











Choice of Cut-off Values

- MLSM :

  Efficient strategy to reduce comput complexity of LSM

- But h**ow do we choose cut-off values ?**

  Cut-off values :
  Sensitive to the noise in the data ;
  Sensitive to the number of the obstacles ;
  Sensitive to the sizes of the obstacles

  No deterministic strategies to determine the cut-off values ;
  Mostly : by experience, or by trial and error

- Li-Liu-Zou, SISC 10

Strengthened Linear Sampling Method with a Reference Object

Main Ideas (Li-Liu-Zou, SISC 09)

D : the unknown scatterer,

1. Introduce an artificial scatterer B, with its shape, position and physical property all known
2. Measure far-field data assoc. with the combined scatterer $D \cup B$
3. Solve the far-field equation at each grid point
4. Find the best cut-off value that fits the boundary of B
5. Use the cut-off value from B to determine D

Two Important Issues

- For SLSM to work, it is natural to require

  interaction between *D* and *B* can not be too weak

- To realize this,

  1. B should not be too small in size compared to D

  2. B should not be too far away from D

- These can be justified mathematically.

B should not be too small or too far

Theorem

B is a reference ball & $\mathrm{dist}(\mathrm{B},\mathrm{D}) > 0$. Then

$$u_\infty(\mathrm{D}\cup\mathrm{B}) = u_\infty(\mathrm{D}) + \mathcal{O}(r) \quad as\ r \to 0$$

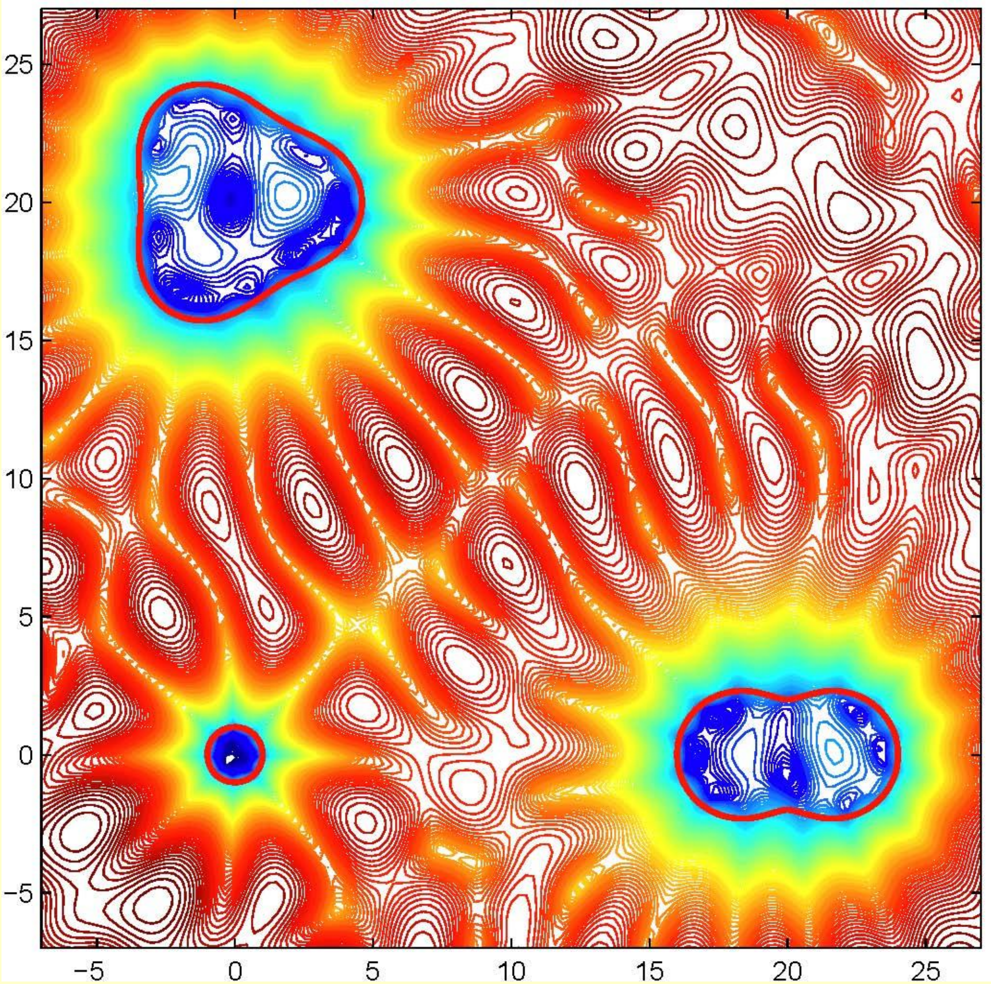
Theorem

B is a reference ball, and $\mathrm{dist}(\mathrm{B},\mathrm{D}) = \rho$. Then

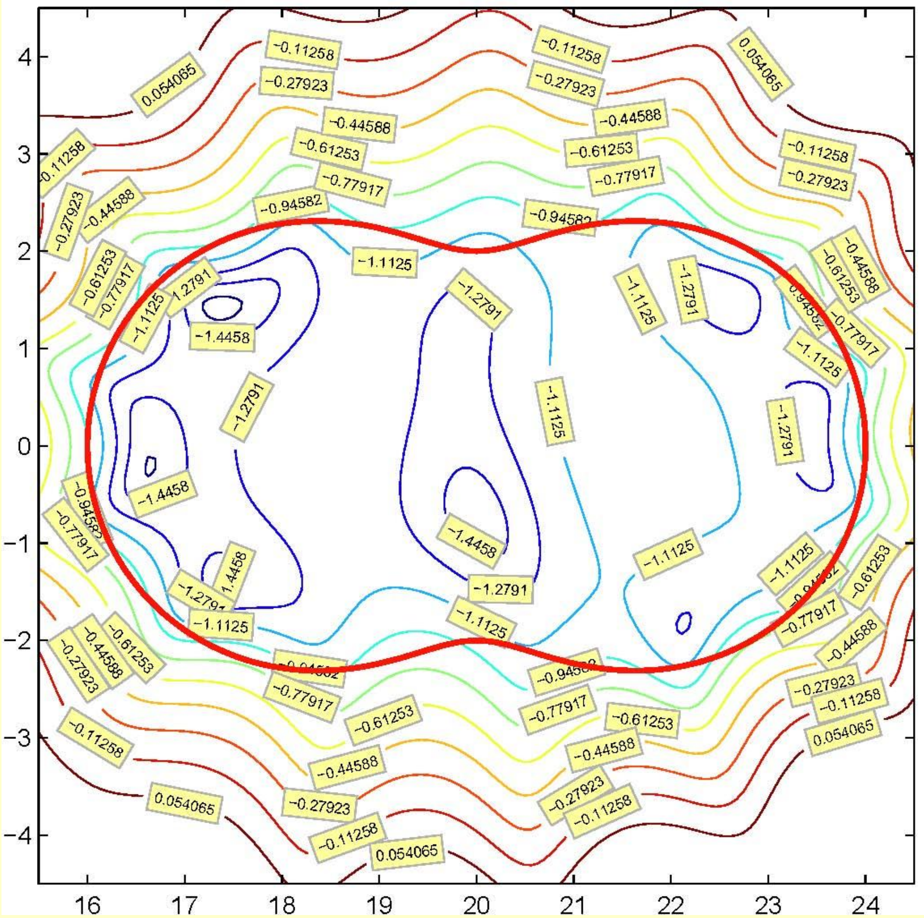
$$u_\infty(\mathrm{D}\cup\mathrm{B}) = u_\infty(\mathrm{D}) + u_\infty(\mathrm{B}) + \mathcal{O}(\rho^{-1})$$

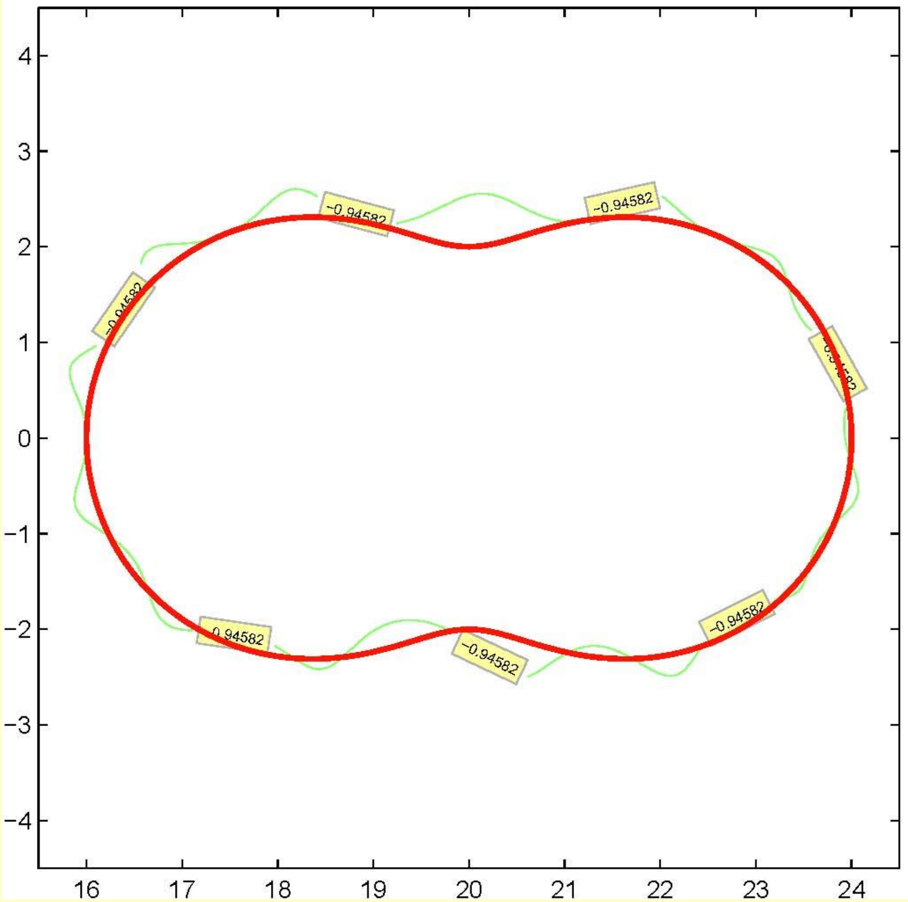
A Scatterer with 2 Objects

A reference ball, a pear displaced at (0,20), and a peanut displaced at (20,0):

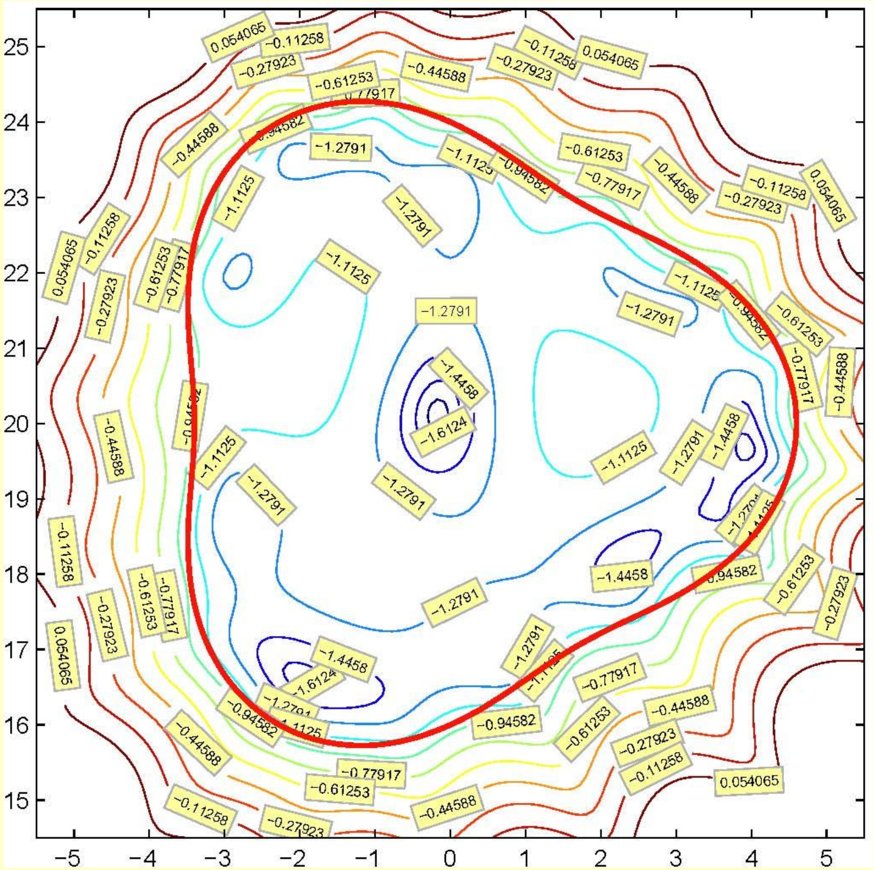


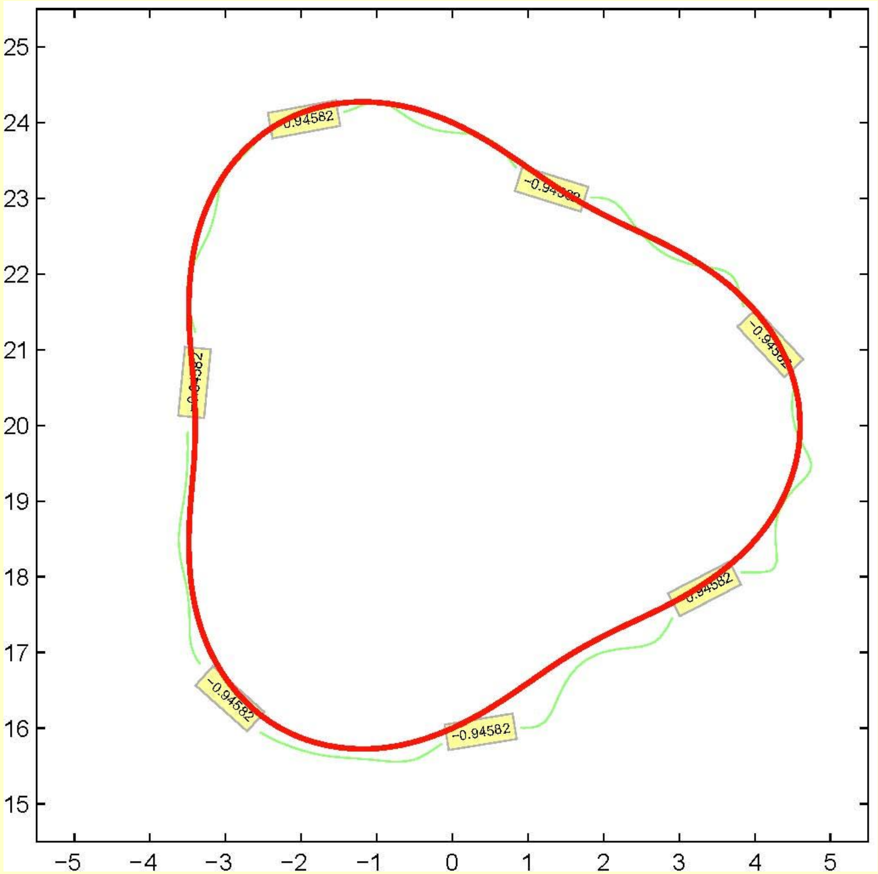
Contour of the peanut





Contour of the pear





Radial Bisection Algorithm

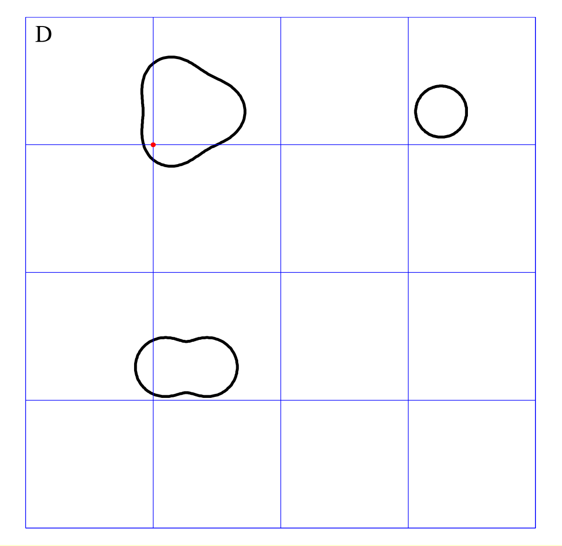
- Simple speed-up: *applicable to all indicator type methods*

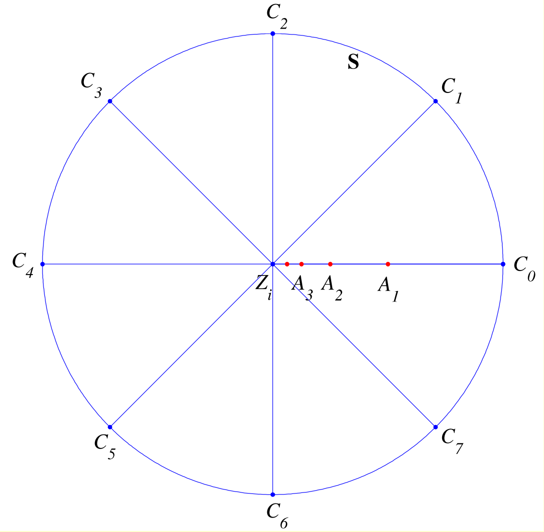
- Interior point algorithm

1 Choose a uniformly coarse mesh;

2 At each grid point, select m radii;

3 On each radius, apply bisection for an interior point





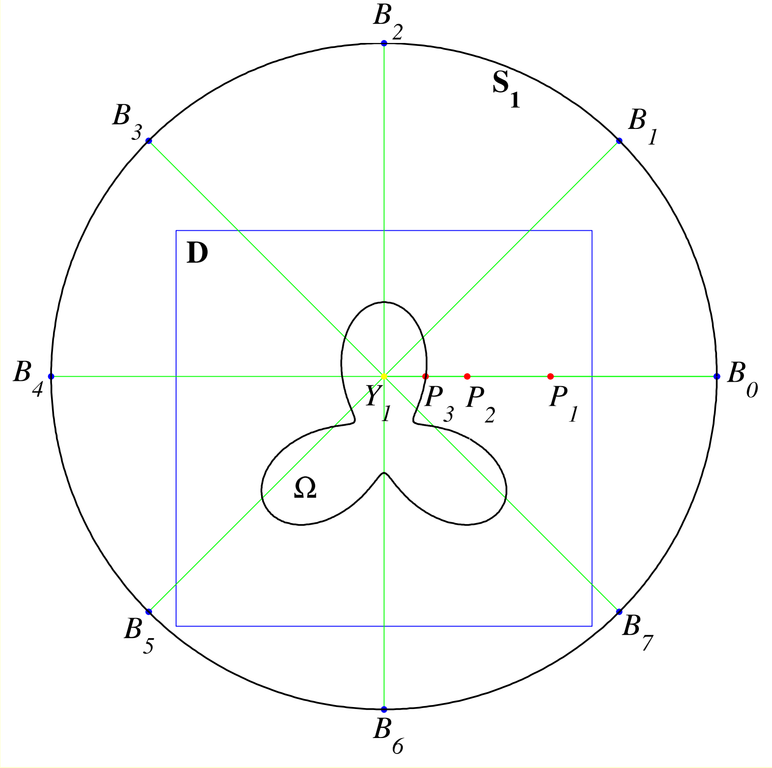
Radial Bisection Algorithm

- Parallel radial bisection algorithm

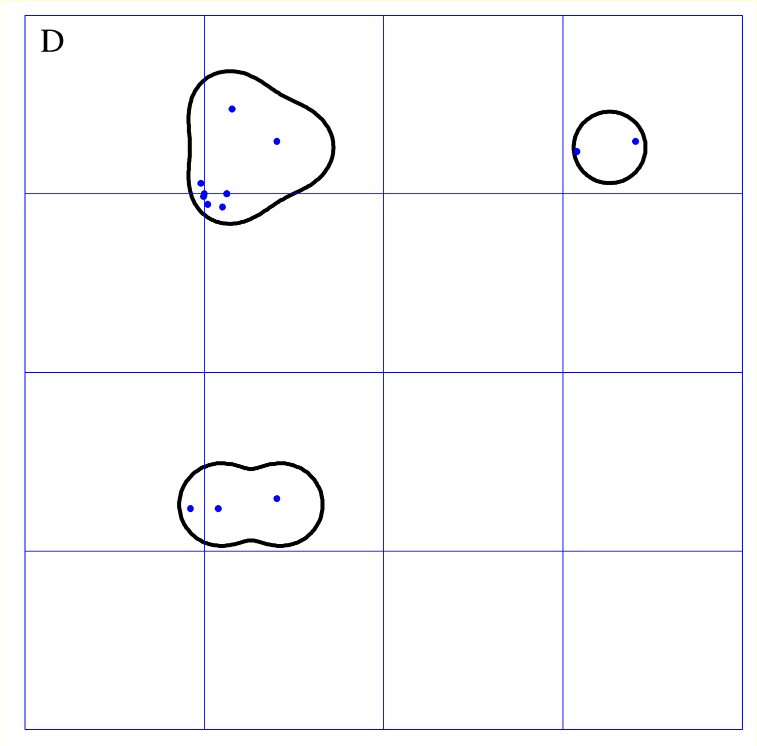
1 At each interior point, select m radii;

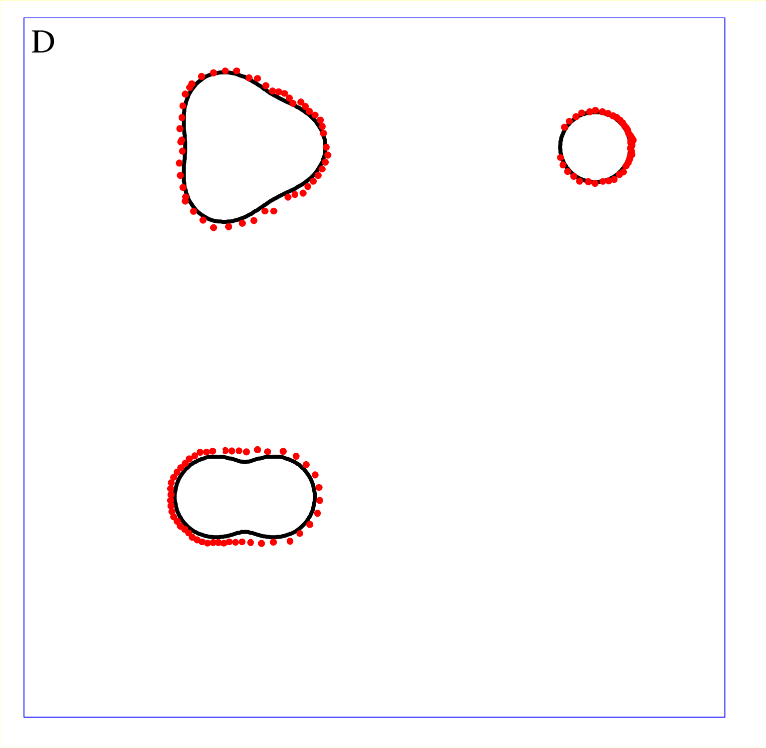
2 On each radius, locate a boundary point;

3 Form the objects using all boundary points

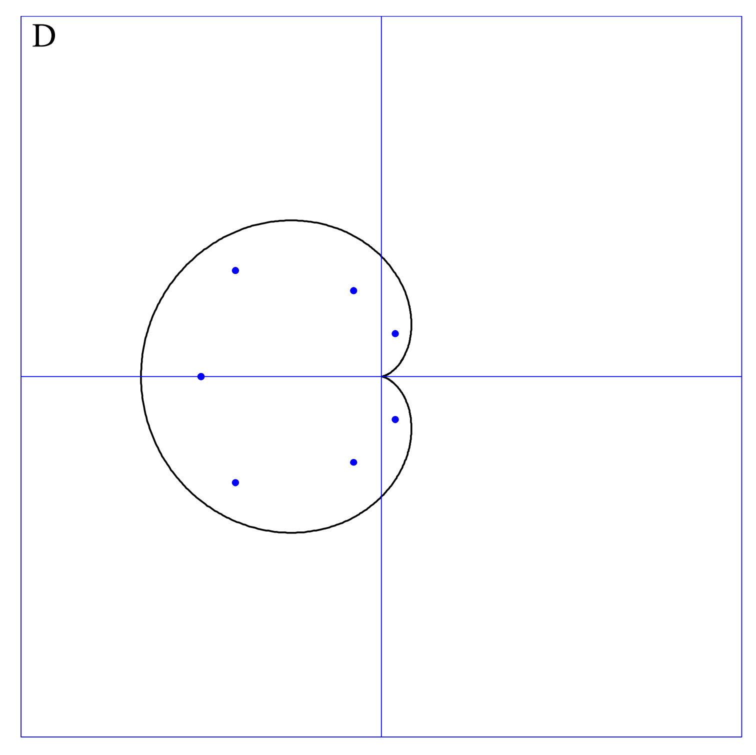


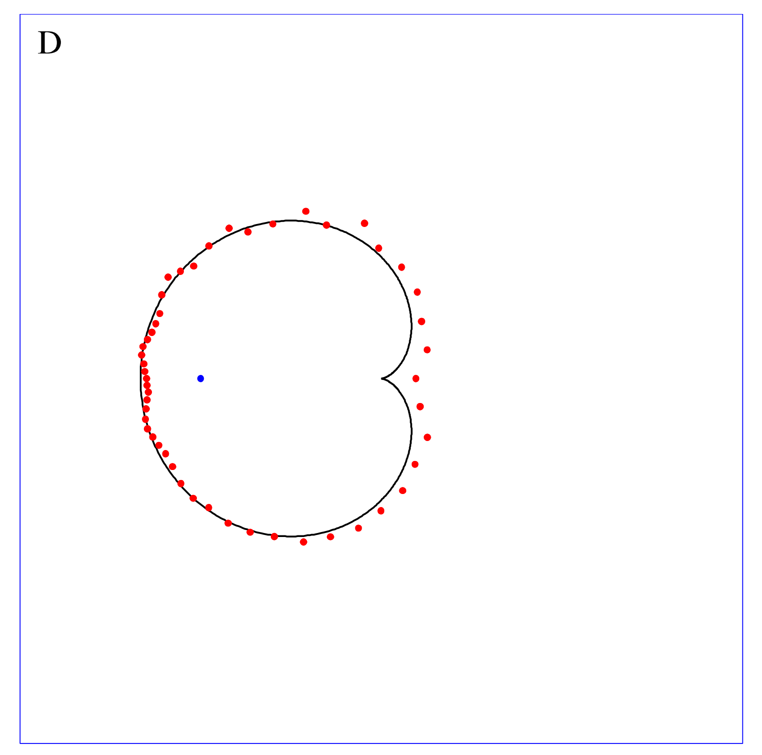
Numerical Experiments





Numerical Experiments





Concluding Part I

MLSM: provides a strategy to reduce the computational complexity of LSM

SLSM: provide a deterministic strategy to choose the cut-off values

Reference Object: up to the practical convenience,

in radar or sonar imaging : place a reference object ;

in medical imaging, geophysical or scientific exploration :

may take objects with known geometry & physical property already inside the scattering system,

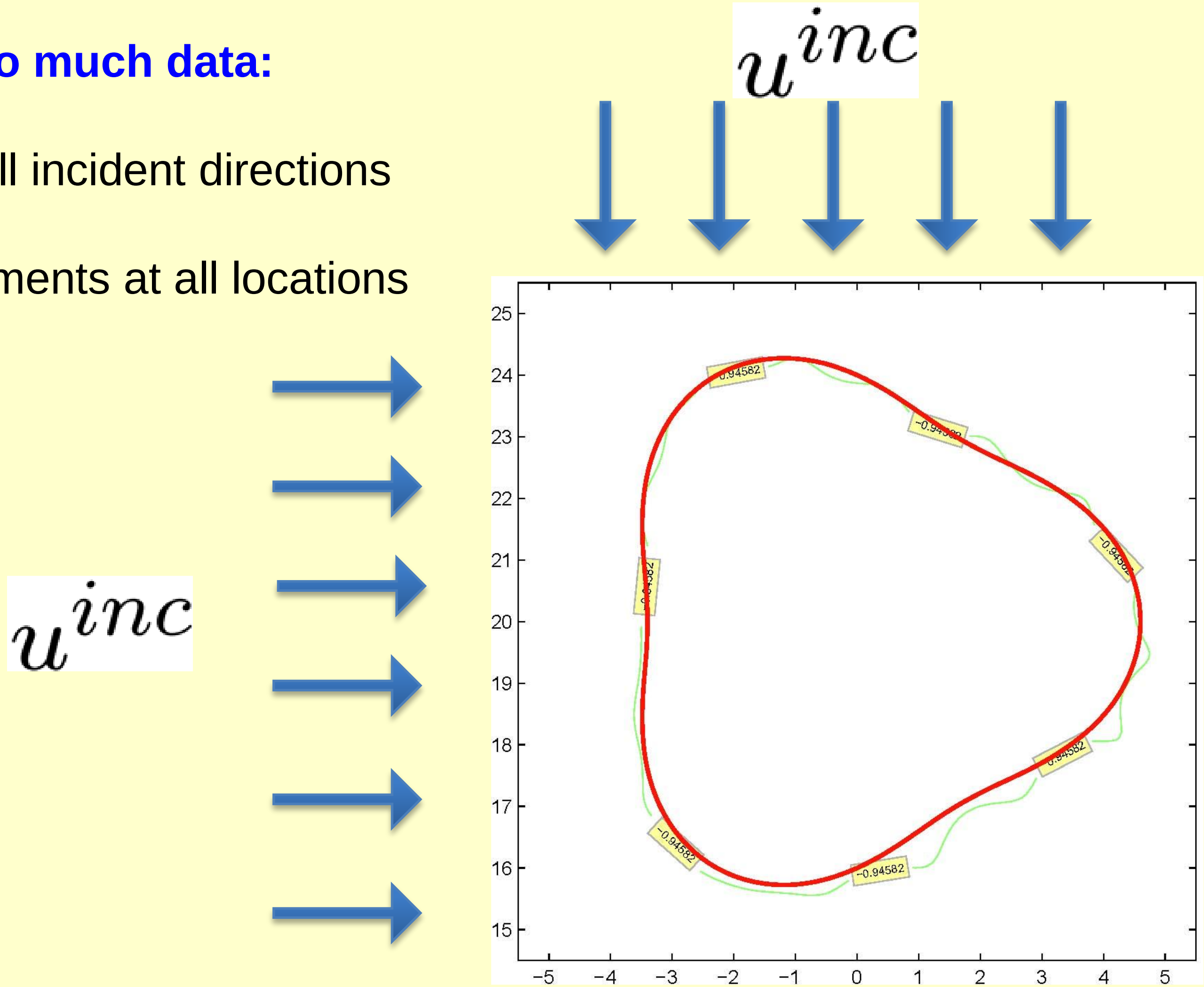
e.g., some object placed in the concerned region before, or some organ inside a patient body

Drawback of LSMs

- **Require too much data:**

  data from all incident directions

  & measurements at all locations



A Direct Sampling Algorithm
(Jin-Ito-Zou, 2011)

- Acoustic, TM or TE model:

$$\Delta u + k^2 n^2(x) u = 0$$

- Total field:

$$\begin{aligned} u &= u^i + u^s \\ &= u^i + \int_\Omega k^2(n^2(x) - 1)\, u\, G(x, y) dy \end{aligned}$$

Derivation of Direct Sampling Algorithm

- Fundamental solution *G* :

$$G(x_p, x_q) - \overline{G}(x_p, x_q) = \int_\Gamma \left[\overline{G}(x, x_q)\partial_n G(x, x_p) - G(x, x_p)\partial_n \overline{G}(x, x_q) \right] ds$$

- Using the radiation condition :

$$\int G(x, x_p)\overline{G}(x, x_q) ds \approx k^{-1} \mathrm{Im}(G(x_p, x_q))$$

Derivation of Direct Sampling Algorithm

- Using the radiation condition :

$$\int G(x,x_p)\overline{G}(x,x_q)ds \approx k^{-1}\mathrm{Im}(G(x_p,x_q))$$

- For the scattered field:

$$u^s(x) = \int_{\widetilde{\Omega}} G(x,y)I(y)dy \approx \sum w_j G(x,y_j)$$

$$I = \kappa^2(n^2(x)-1)\,u$$

- From the above two :

$$\int_{\Gamma} u^s(x)\,\overline{G}(x,x_p)\,ds \approx k^{-1}\sum w_j\,\mathrm{Im}(G(y_j,x_p))$$

A Direct Sampling Algorithm
(Jin-Ito-Zou, 2011)

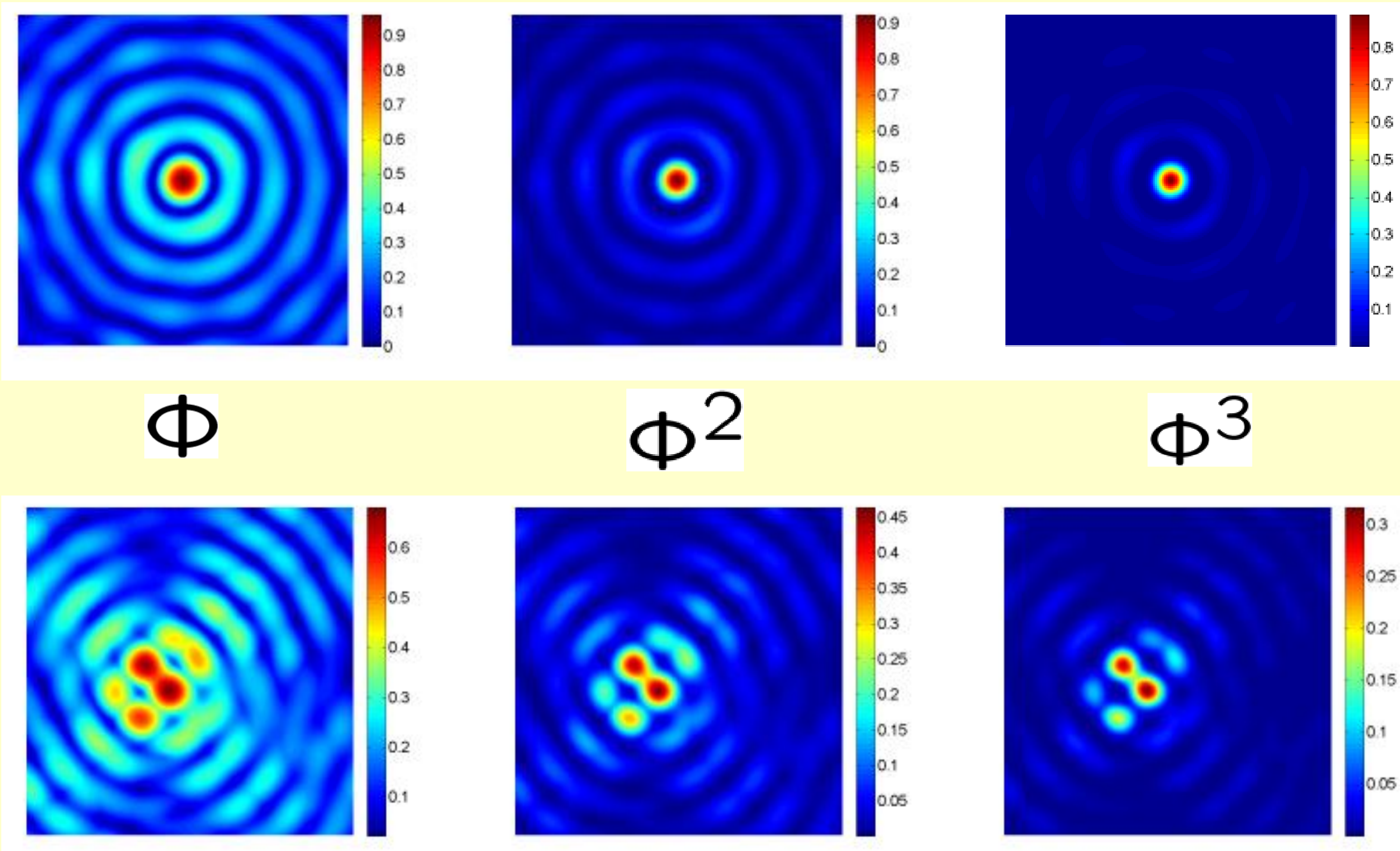
- Recall

$$\int_\Gamma u^s(x)\,\overline{G}(x,x_p)\,ds \approx k^{-1}\sum w_j\,\mathrm{Im}(G(y_j,x_p))$$

- Index func for support of inhomog. media :

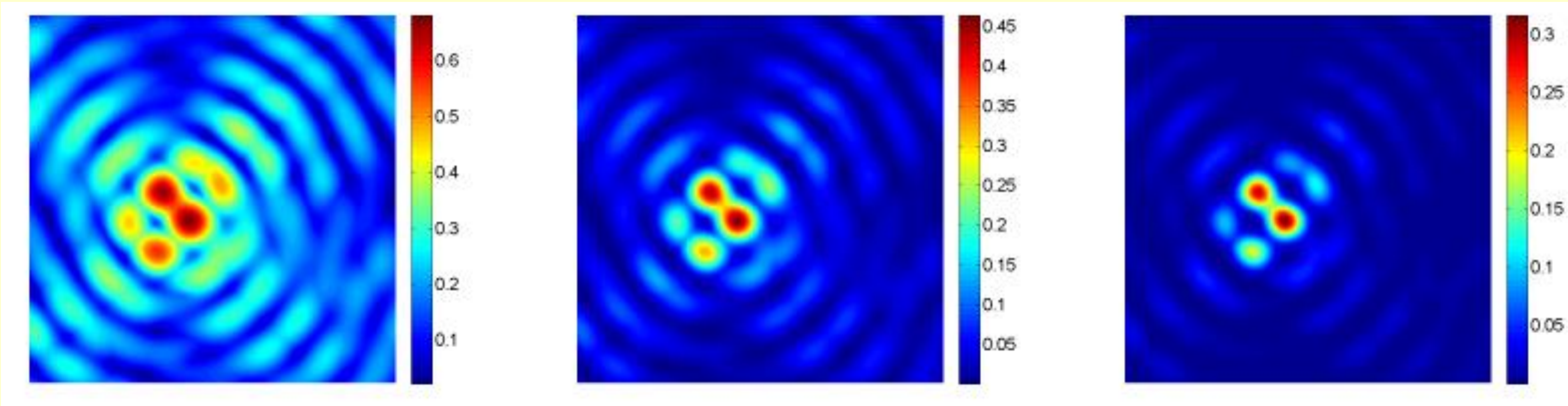
$$\Phi(x_p) = \frac{|\langle u^s, G(\cdot,x_p)\rangle_\Gamma|}{\|u^s\|\,\|G(\cdot,x_p)\|}$$

Numerical Examples I

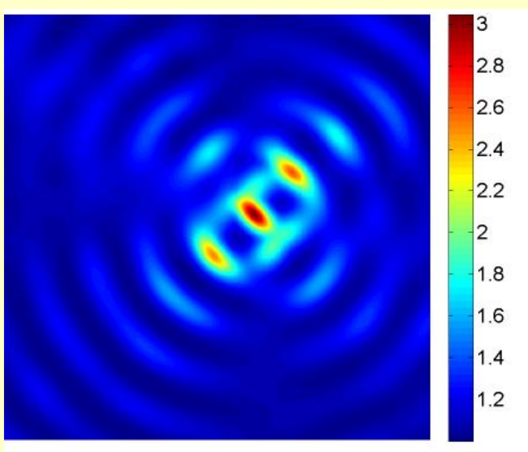


Two incidents: 20% noise

Comparison with MUSIC



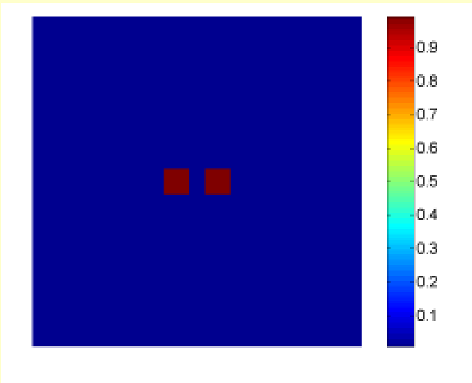
Φ Φ^2 Φ^3



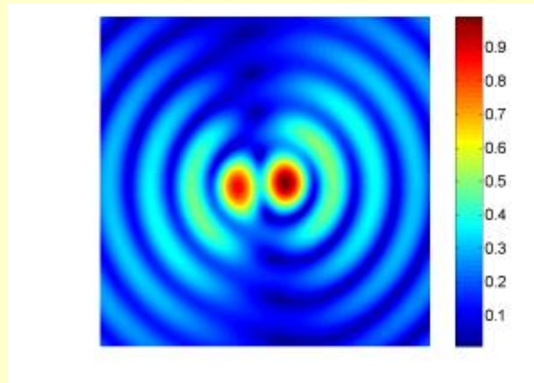
Reconstruction by MUSIC

Two incidents: 20% noise

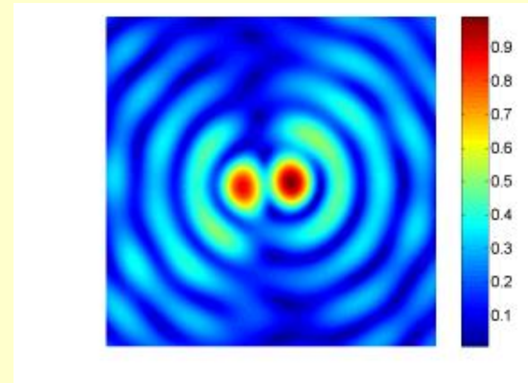
Numerical Examples II



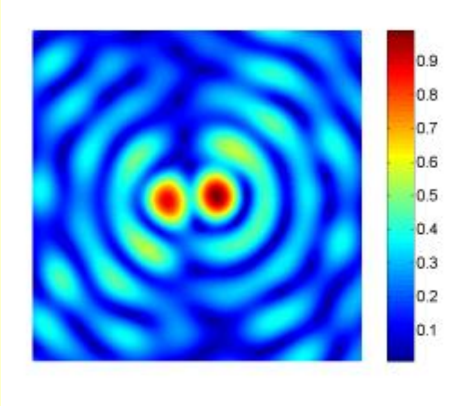
(a) true scatterer



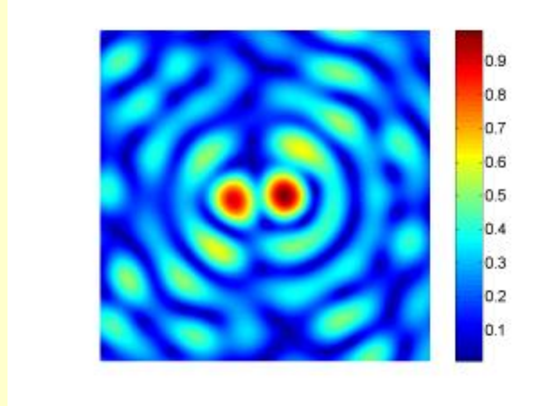
(b) exact data



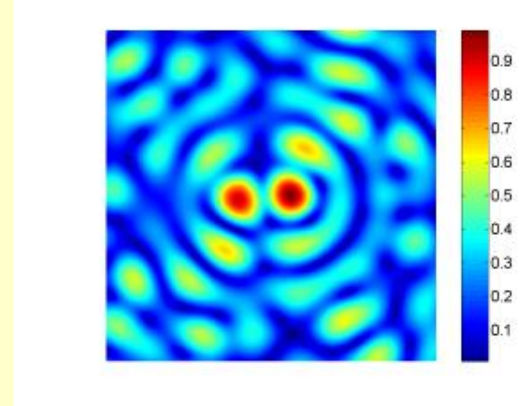
(c) $\varepsilon = 10\%$



(d) $\varepsilon = 20\%$



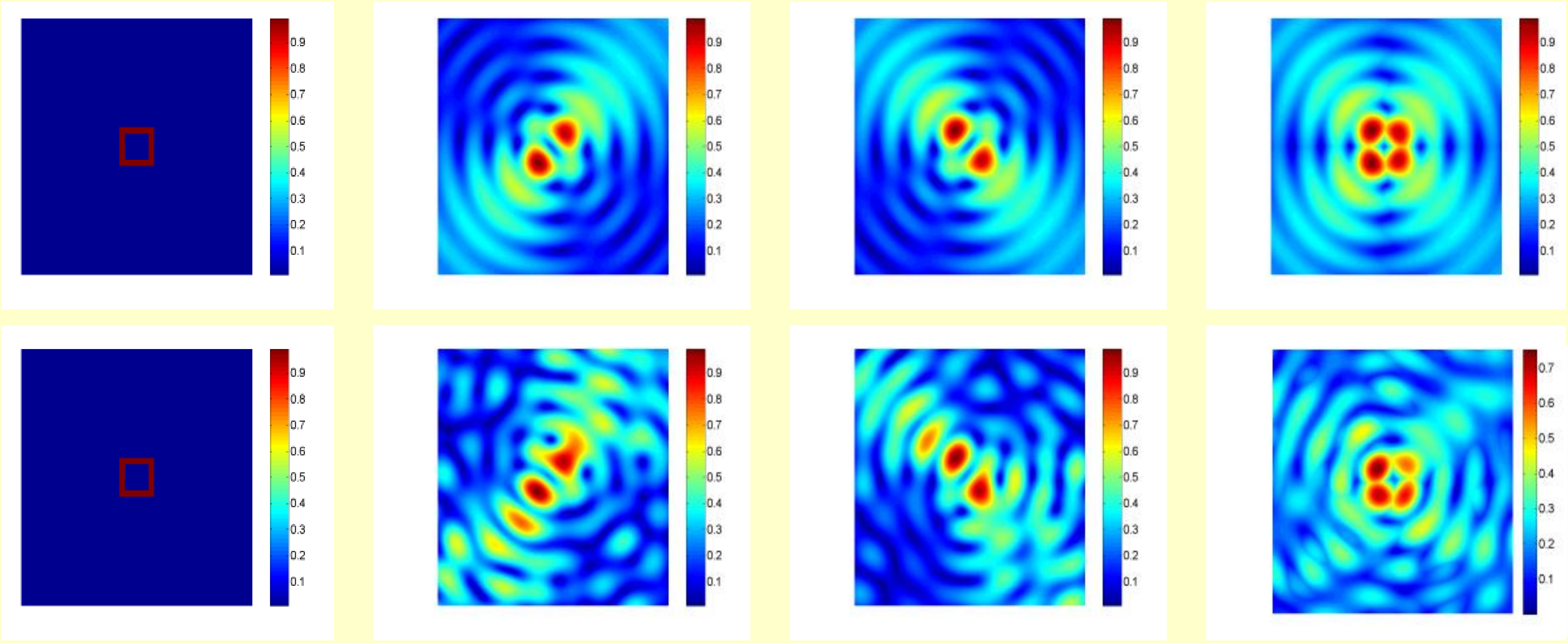
(e) $\varepsilon = 30\%$



(f) $\varepsilon = 40\%$

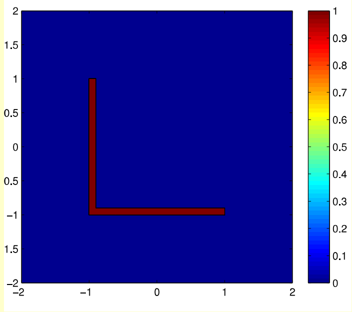
One incident at (1, 1)

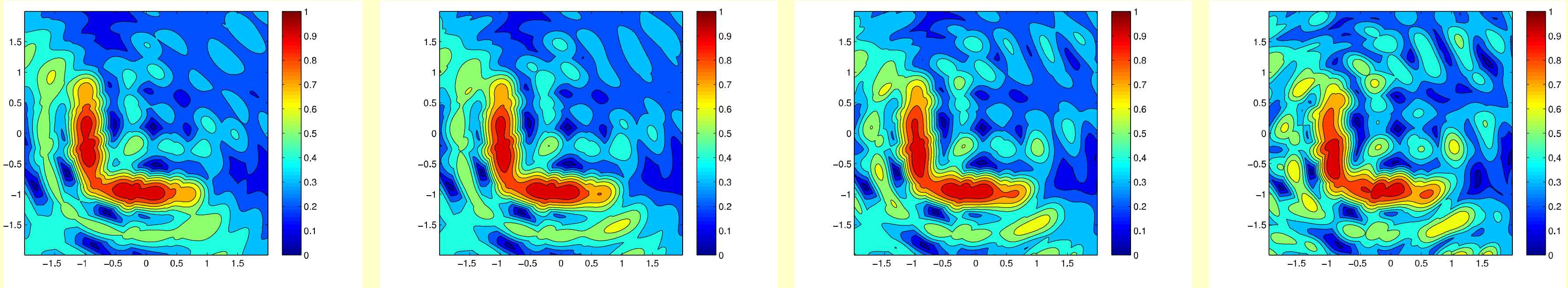
Numerical Examples III



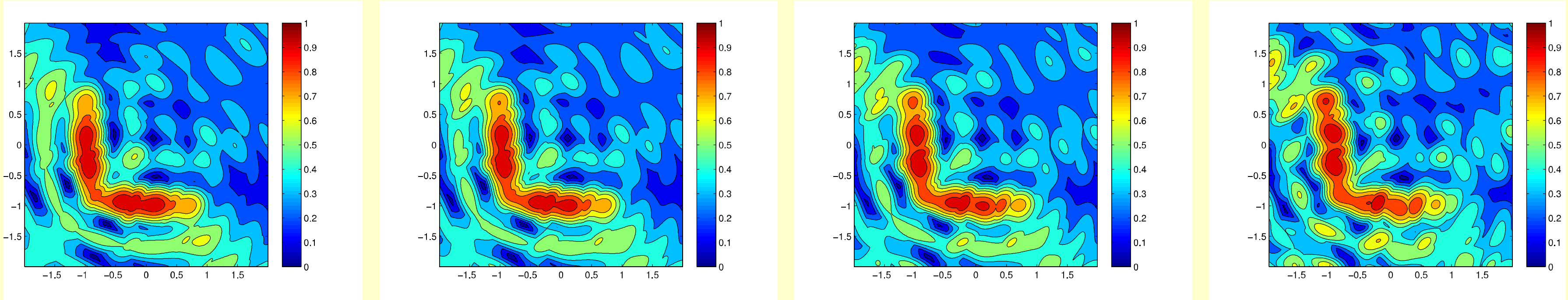
(a) true scatterer (b) $d_1 = \frac{1}{\sqrt{2}}(1,1)^{\mathrm{T}}$ (c) $d_2 = \frac{1}{\sqrt{2}}(1,-1)^{\mathrm{T}}$ (d) d_1, d_2

Numerical Examples III





DSM(n) using two incident waves: from left to right, noise level is 0, 5%, 10%, 20%.



DSM(f) using two incident waves: from left to right, noise level is 0, 5%, 10%, 20%.

A Multilevel Sampling Algorithm

(Liu-Zou, 2013)

- Acoustic, TM or TE model:

$$\Delta u + k^2 (\chi(x) + 1) u = 0$$

- Total field:

$$u = u^i + \int_\Omega k^2 G(\cdot, y)\, \chi(x)\, u(x)\, dy$$

- Introduce the induced current : $w = \chi\, u$

$$w = \chi\, u^i + \chi\, G_D(w) \quad \text{in} \quad D$$

$$u^s = G_S(w) \quad \text{in} \quad S$$

A Multilevel Sampling Algorithm
(Liu-Zou, 2013)

- Back-propagation:

$$\min \|u^s - G_s(w)\|^2 \quad \text{over} \quad \{ G_s^*(u^s) \}$$

- Approximate contrast source:

$$w_j = \lambda_0 G_S^*(u_j^s) \quad \forall\, j$$

- Approximate the contrast value :

$$\min_{\chi(\boldsymbol{x})} \sum_j \left| \Big(\chi u_j^i - w_j + \chi G_D(w_j) \Big)(\boldsymbol{x}) \right|^2$$

- Approximate the contrast value :

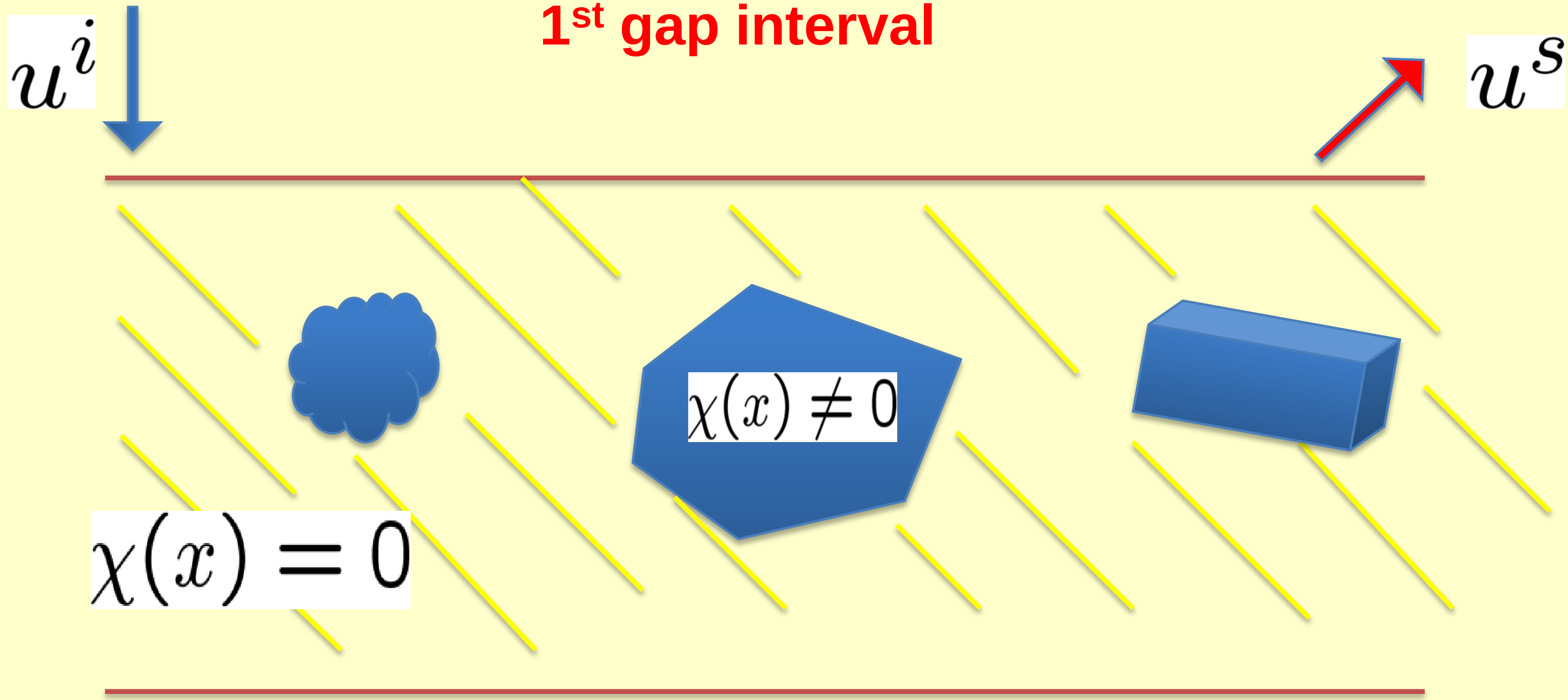
$$\chi(\boldsymbol{x}) = Re(\cdots) \quad \forall\, \boldsymbol{x} \in D$$

Reconstruct ***Shape, Location & Physics***

- **Arrange the contrast values at all sampling points:**

$$\chi_1 < \cdots < \chi_{n_0} < \chi_{n_0+1} < \cdots < \chi_m$$

1st gap interval



A Multilevel Sampling Algorithm

- Give a sampling domain D_0
- Approximate contrast source by back-propagation :

$$w_j = \lambda_0 \, G_S^*(u_j^s) \quad \forall \, j$$

- Approximate the contrast value :

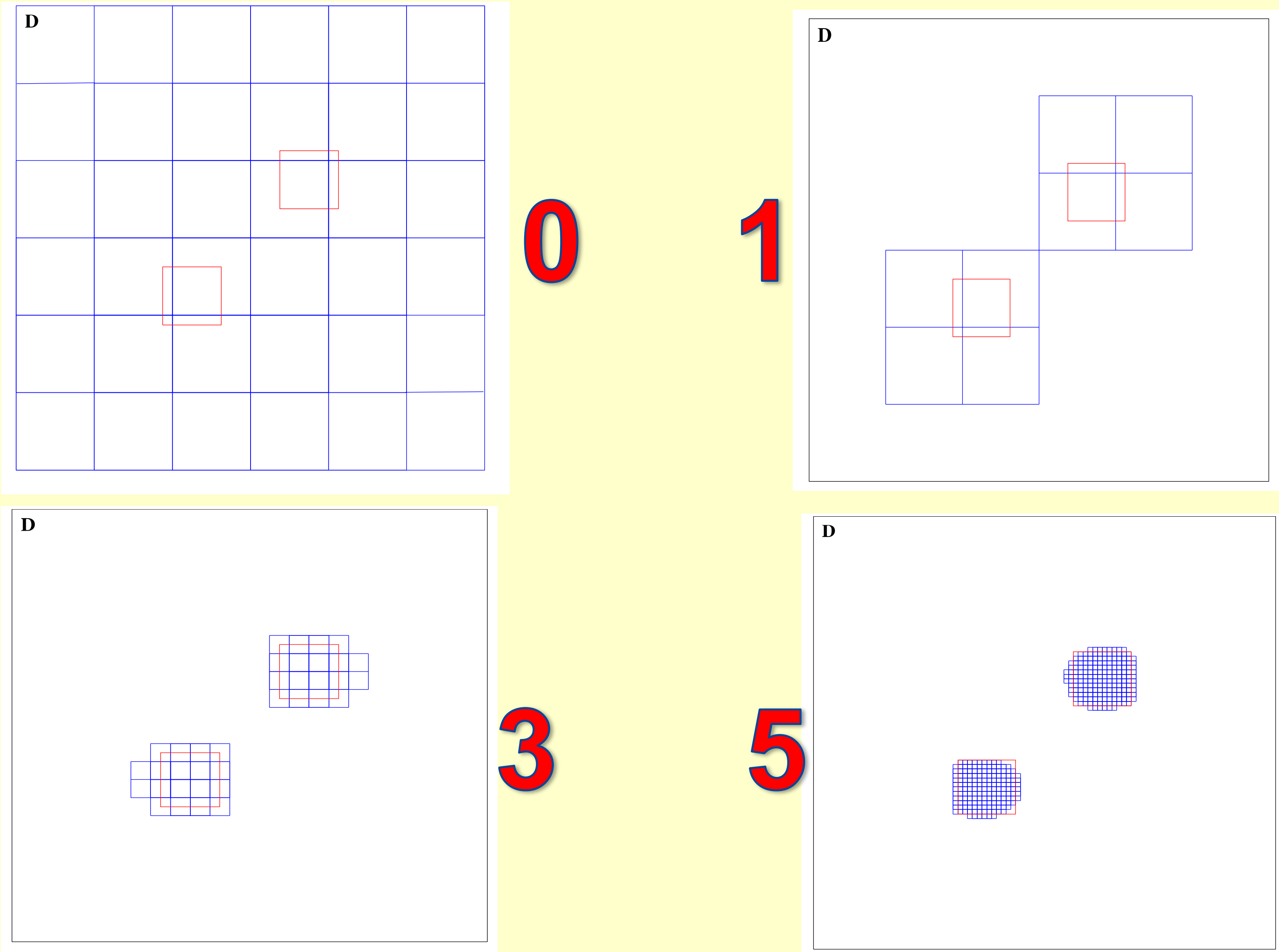
$$\min_{\chi(\boldsymbol{x})} \quad \sum_j \left| \left(\chi u_j^i - w_j + \chi G_D(w_j) \right)(\boldsymbol{x}) \right|^2$$

- Arrange the contrast values & find the 1st gap interval:

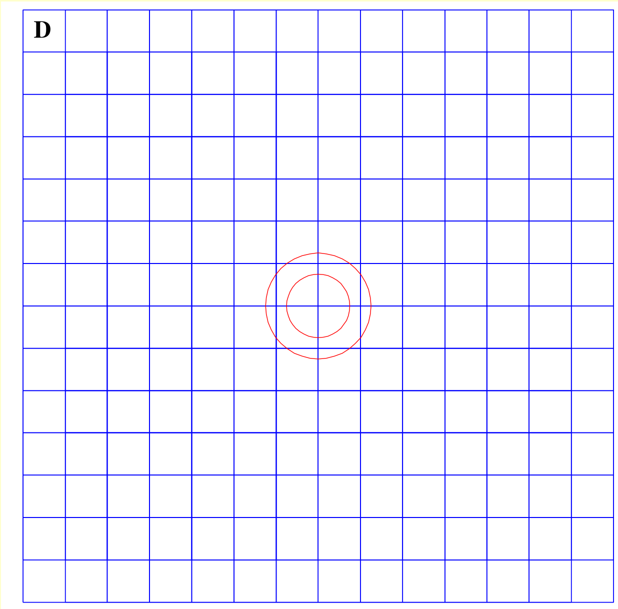
$$\chi_1 < \cdots < \chi_{n_0} < \chi_{n_0+1} < \cdots < \chi_m$$

- Drop all sampling points with small values: D_1
- Repeat

Numerical Examples

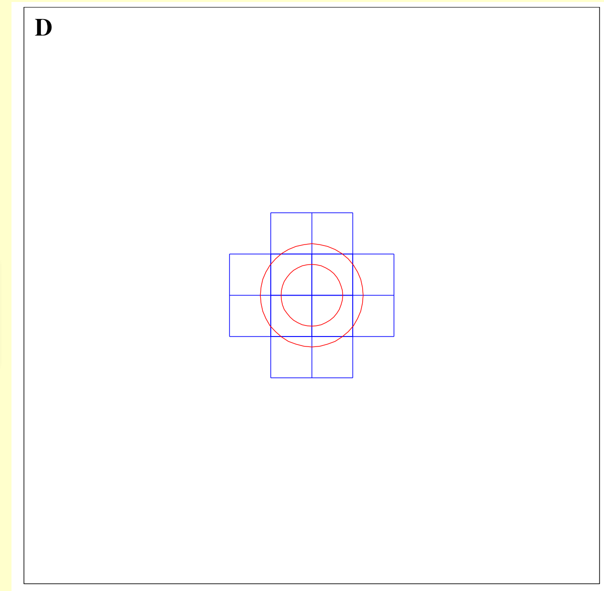


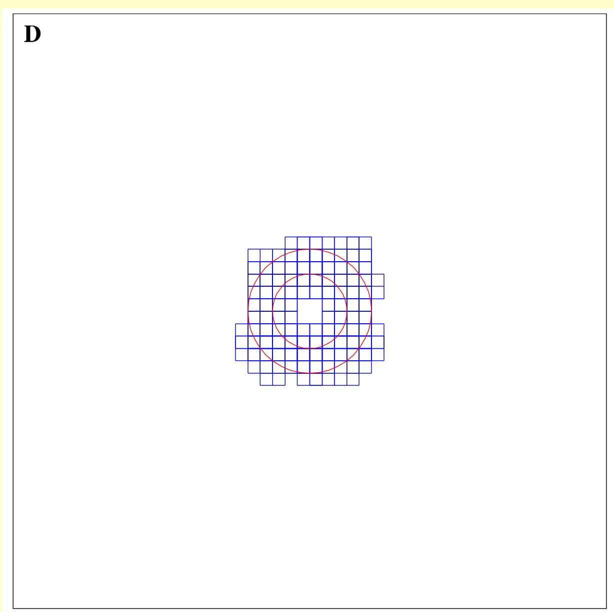
Numerical Examples



0

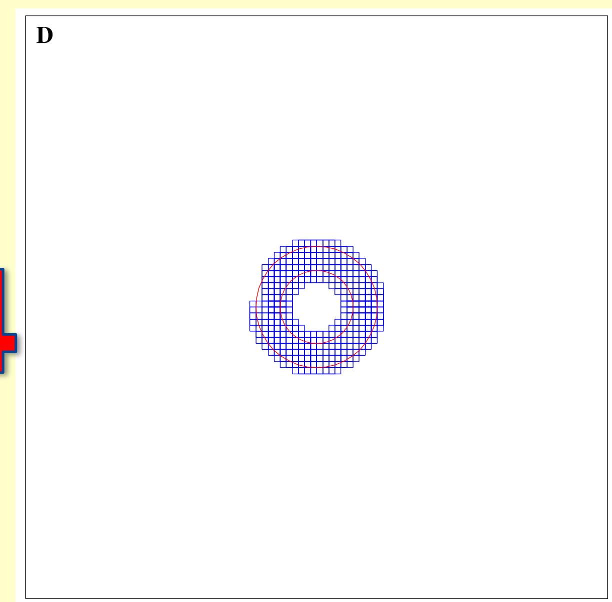
1



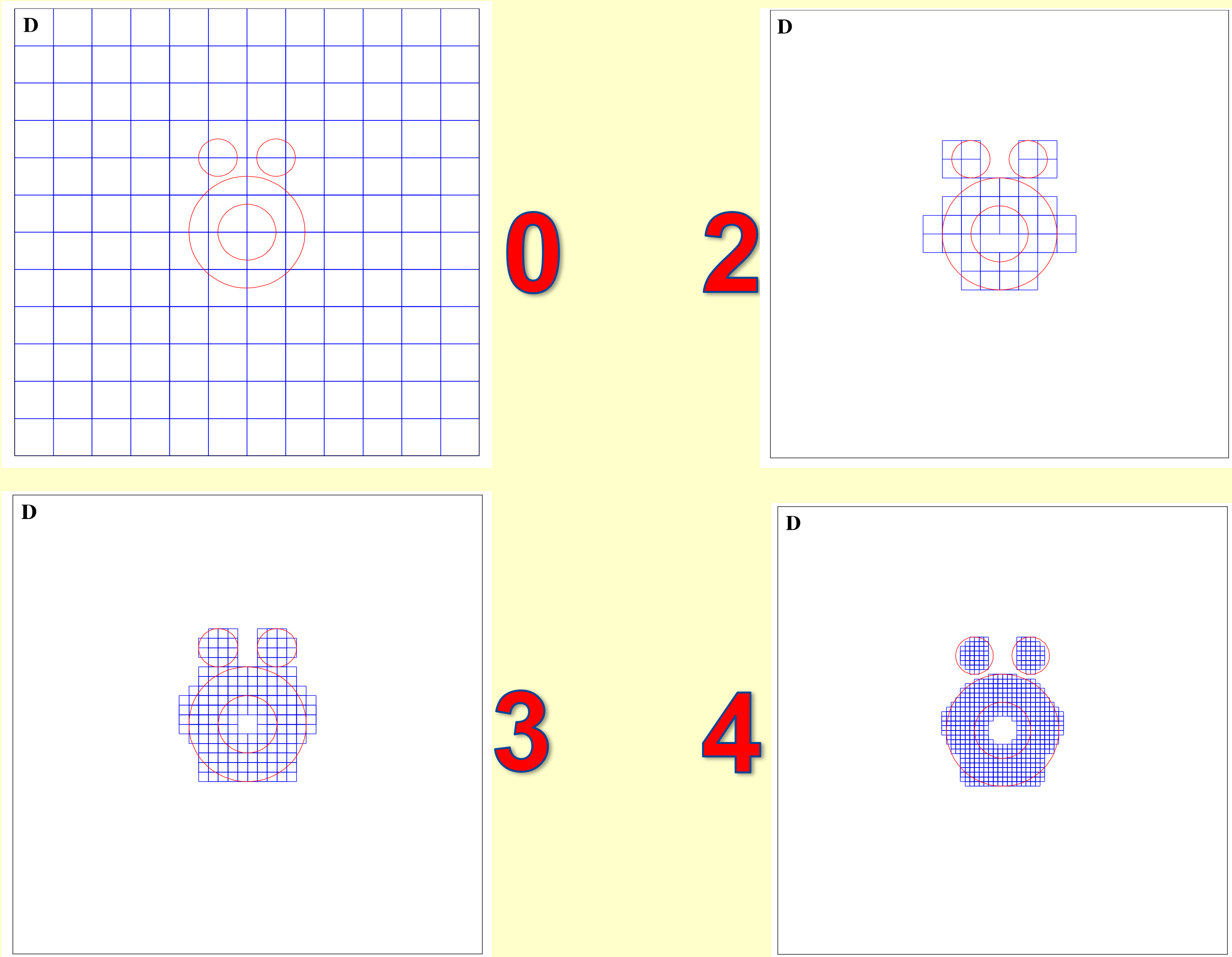


3

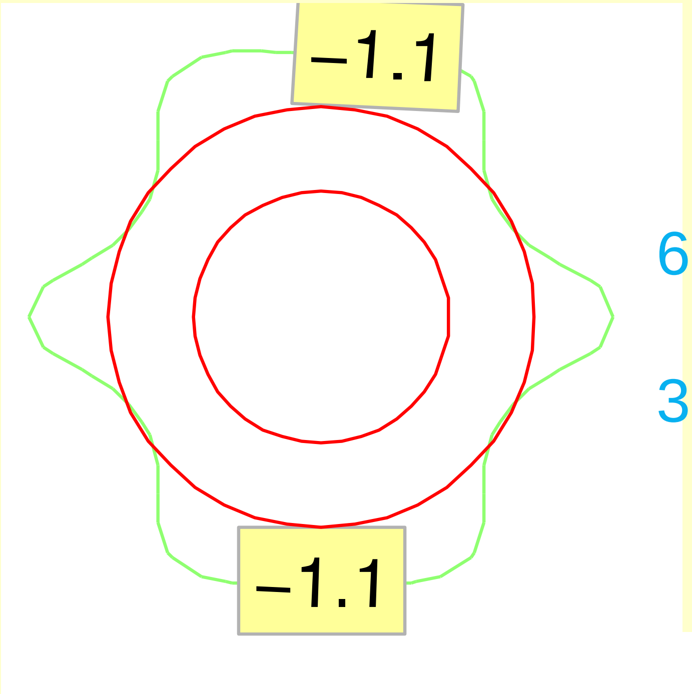
4



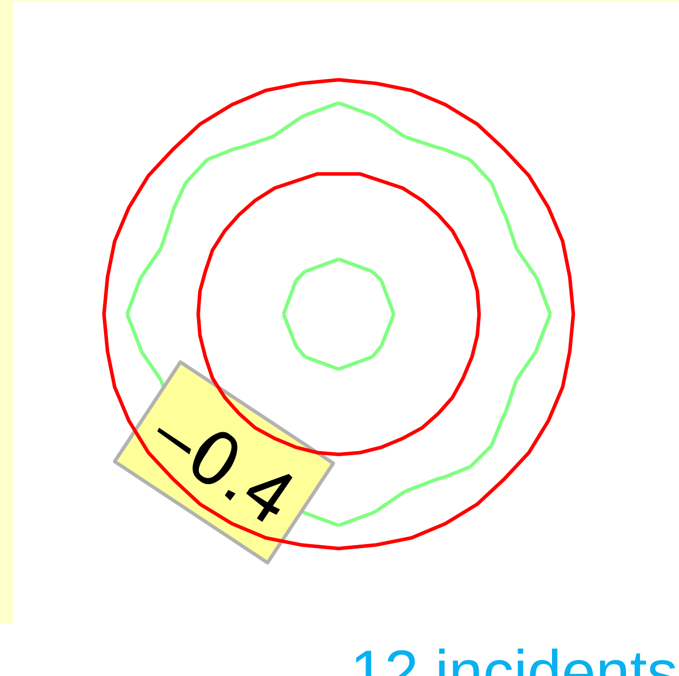
Numerical Examples



Comparisons with LSM

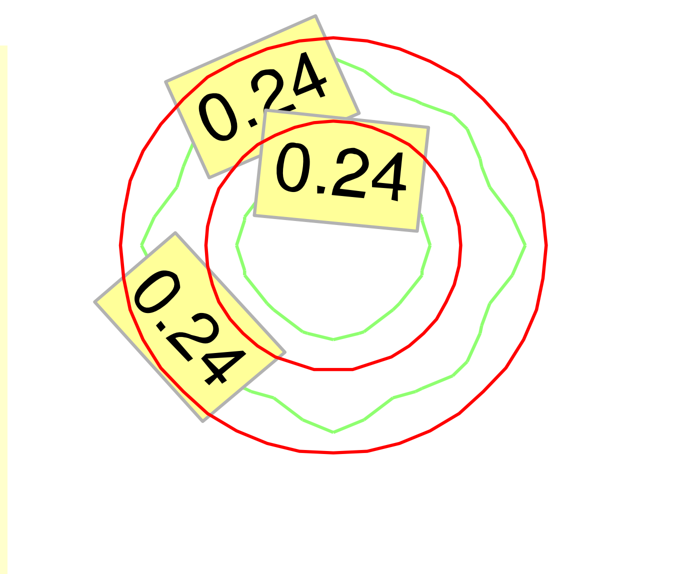


6 incidents
&
30 receivers

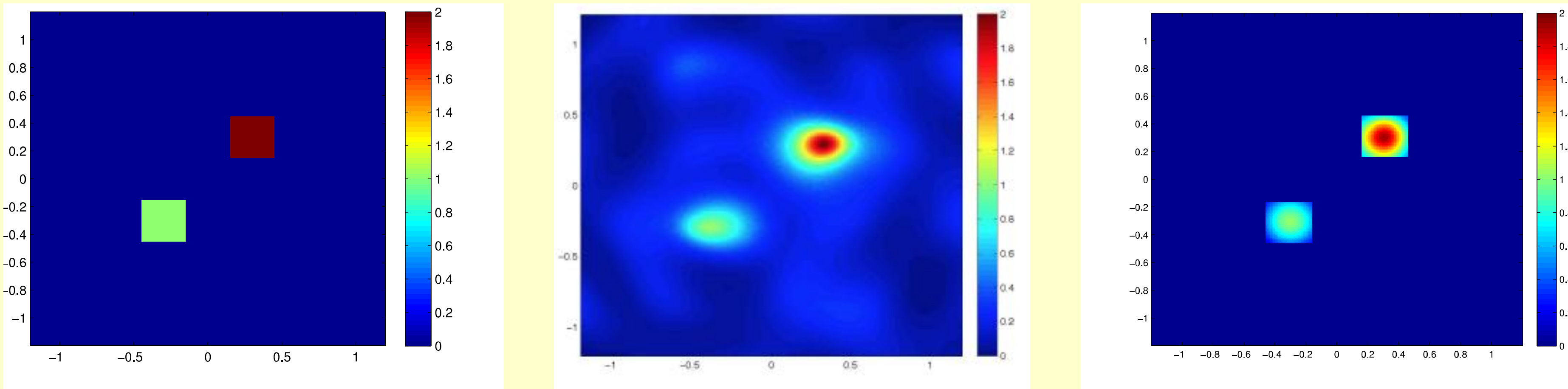


12 incidents
&
30 receivers

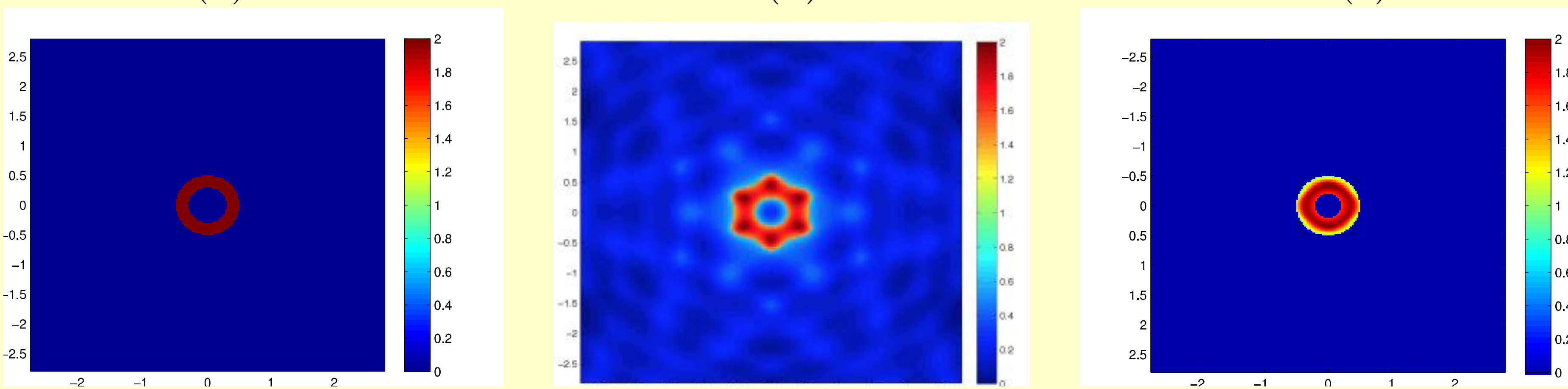
36 incidents
&
30 receivers



Numerical Example

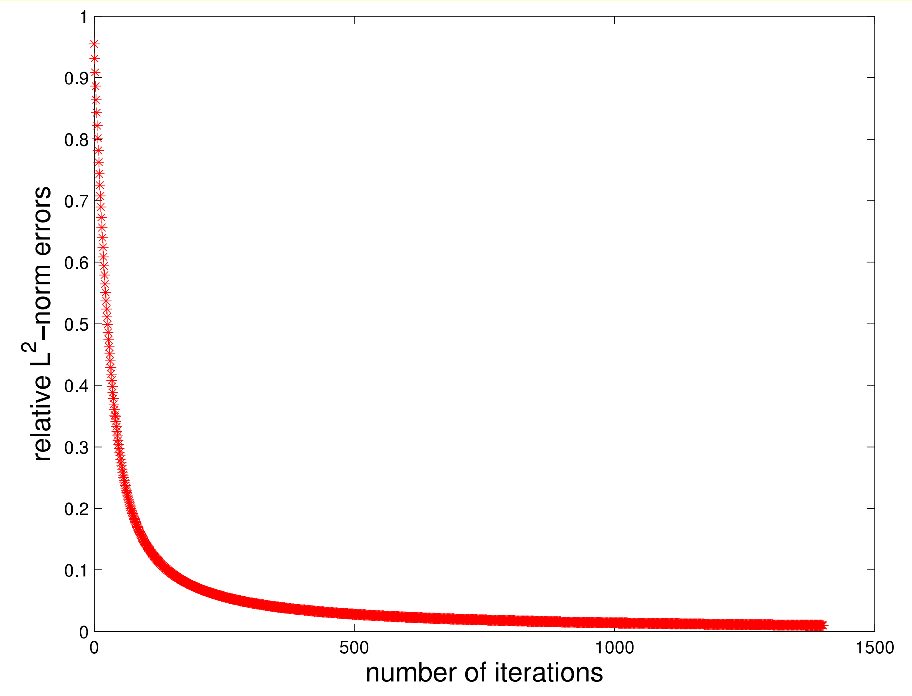


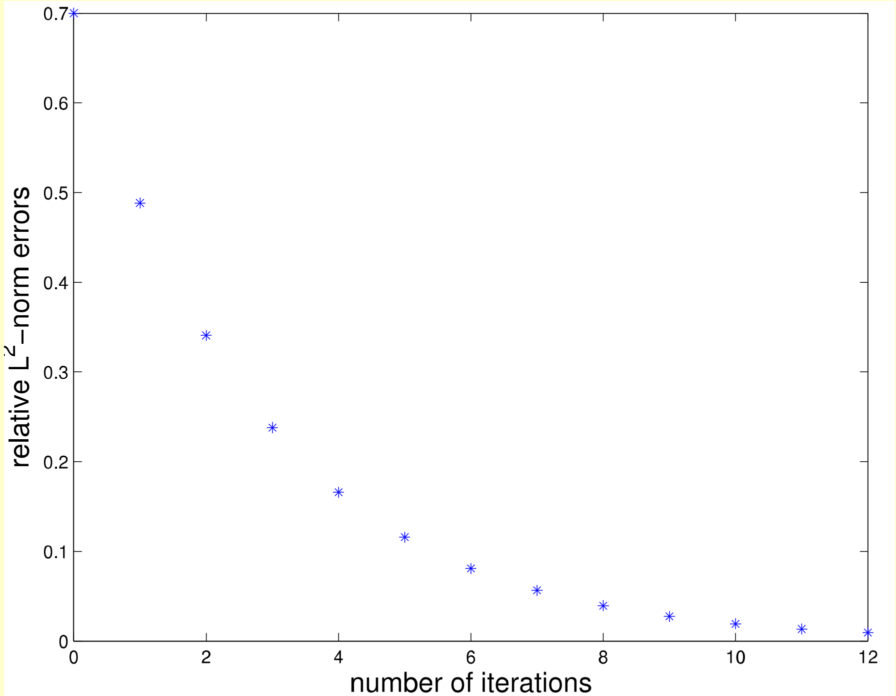
(a) (b) (c)



With h=0.015, smallest for CSI method

Convergence history





Semi-smooth Newton with Sparsity
(Jin-Ito-Zou, 2012)

- Index func $\Phi(x_p)$ for inhomogeneous media
- Using estimated medium *D* & inhomog. η :

$$K\eta \equiv \int_D G(x,y)\hat{u}(y)\eta(y)dy = u^s(x) \quad \forall x \in \Gamma$$

- Minimization with sparsity :

$$\min \frac{1}{2}\int_\Gamma |K\eta - u^s|^2 ds + \alpha \int_D |\eta| dx + \frac{\beta}{2}\int_D |\nabla \eta|^2 dx$$

- L1: preserve sparsity, localized, keep clean background;
  alone: instable, too spiky, no groupwise structure

H1: globally smooth, overly diffusive, blurry background;

Semi-smooth Newton with Sparsity

- Minimization with sparsity :

$$\min \frac{1}{2}\int_{\Gamma} |K\eta - u^s|^2 ds + \alpha \int_D |\eta| dx + \frac{\beta}{2}\int_D |\nabla \eta|^2 dx$$

- Equiv to solving highly nl variational system :

$$K^*K\eta - \beta\Delta\eta - K^*u^s \in -\alpha\partial\psi(\eta)$$

- Equiv to solving the variational system :

$$K^*K\eta - \beta\Delta\eta - K^*u^s + \alpha\lambda = 0$$

$$\lambda - \frac{\lambda + c\eta}{\max(1, |\lambda + c\,\eta|)} = 0$$

Semi-smooth Newton Algorithm

- Initialize η^0, λ^0; set $c > 0$, $k = 0$
- Active set $\mathcal{A}^k$, inactive set $\mathcal{I}^k$:

$$\mathcal{A}^k = \{x \in D : |\lambda^k + c\,\eta^k| \leq 1\}$$
$$\mathcal{I}^k = \{x \in D : |\lambda^k + c\,\eta^k| > 1\}$$

- Compute $$d^k = |\lambda^k + c\eta^k|, \quad F^k = \frac{\lambda^k + c\,\eta^k}{|\lambda^k + c\,\eta^k|}$$
- Solve for $(\eta^{k+1}, \lambda^{k+1})$ on $\mathcal{I}^k$:

$$K^*K\eta^{k+1} - \beta\,\Delta\eta^{k+1} - K^*u^s + \alpha\,\lambda^{k+1} = 0$$
$$\lambda^{k+1} - \frac{c}{d^k - 1}(I - F^k)\eta^{k+1} - \frac{\lambda^k}{\max(|\lambda^k|, 1)} = 0$$

Nice Feature of Semi-smooth Newton

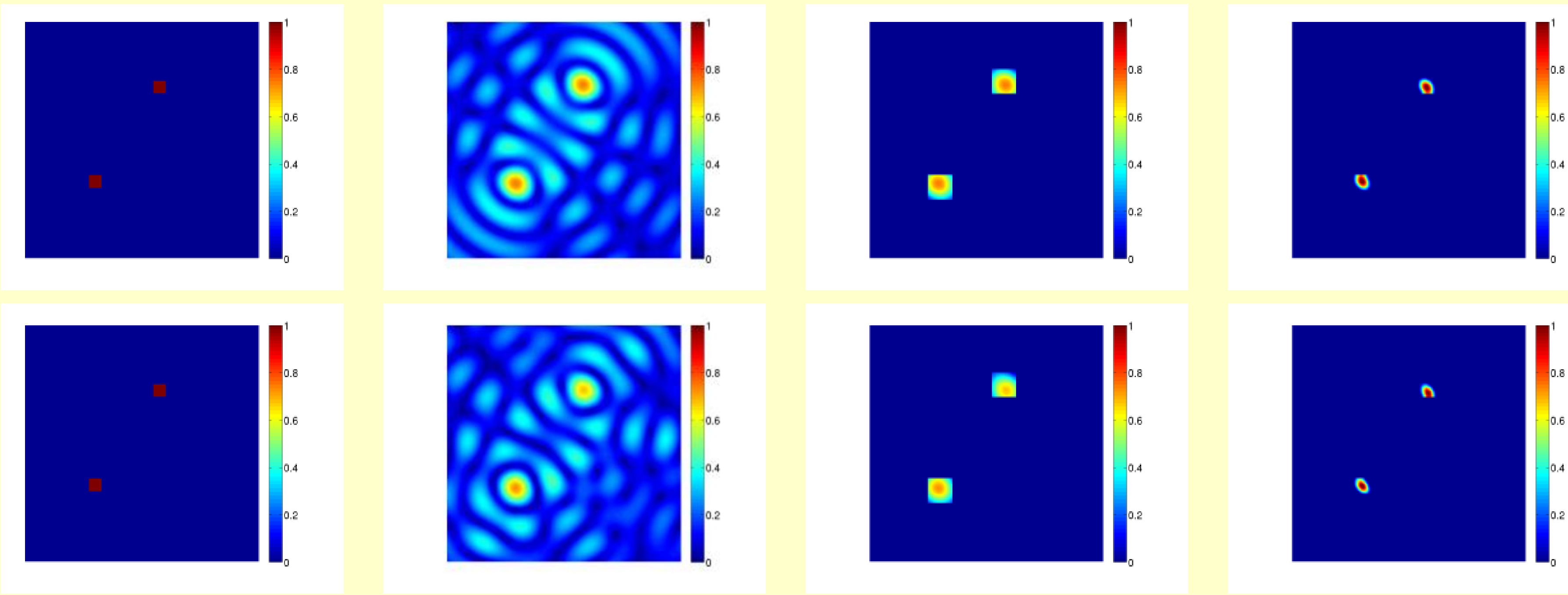
- Major step : solve a linear system on $\mathcal{I}^k$
- As iteration goes on :

  linear system becomes smaller & less ill-conditioned;

  captures more & more refined details of inhomogeneity;

  convergence: rather stable & fast

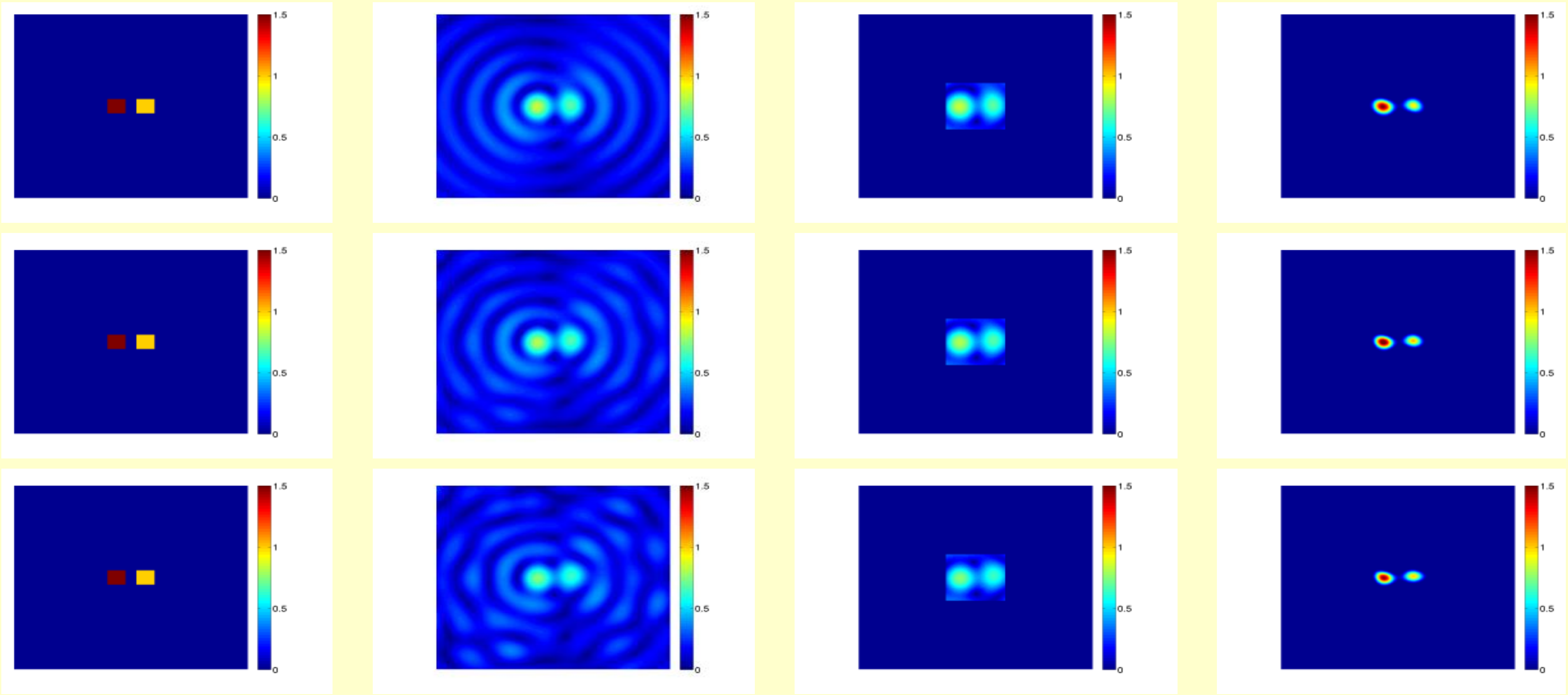
Numerical Example I



(a) true scatterer (b) index Φ (c) index $\Phi|_D$ (d) sparse recon.

One incident at (1, 1): exact data; 20% noise

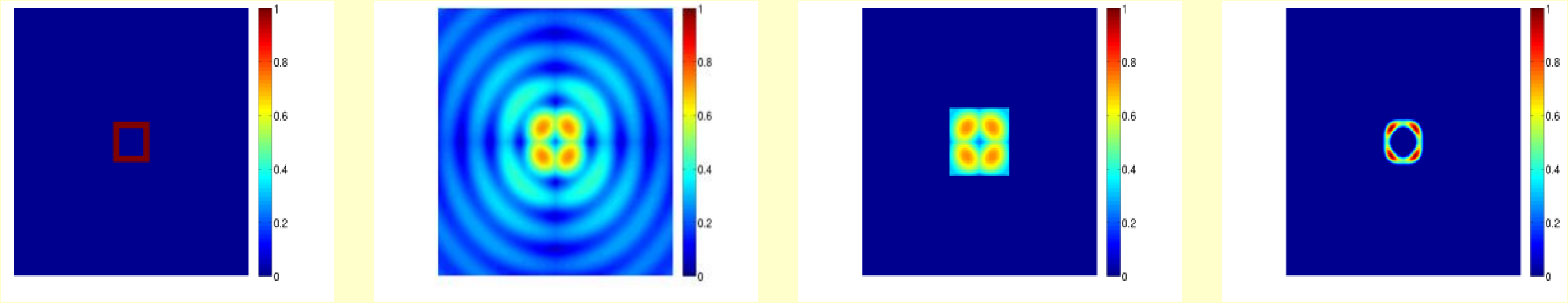
Numerical Example II



(a) true scatterer (b) index Φ (c) index $\Phi|_D$ (d) sparse recon.

One incident at (1, 1): exact data; 10%, 20% noise

Numerical Example III



One incident at (1, 1): exact data; 20% noise

Regularization Parameters

Table 1: Regularization parameters (α, β) for the examples.

example	1(a)	1(b)	2	3
$\epsilon = 0\%$	(2.0e-6,1.5e-9)	(8.0e-6,1.4e-8)	(7.0e-6,1.0e-9)	(2.5e-9,4.0e-14)
$\epsilon = 20\%$	(3.0e-6,2.0e-9)	(8.5e-6,9.0e-9)	(7.0e-6,5.0e-9)	(2.5e-9,5.0e-14)

DSM's Extension to Others

- **Extensions to**

Electromagnetic medium scattering, Ito-Jin-Zou 13;

Electric impedance tomography, Chow-Ito-Zou 14;

Diffusive Optical Tomography, Chow-Ito-Liu-Zou 14;

Moving objects, … …

DSM for EIT

- Electrical Impedance Tomography :

$$\nabla \cdot (\sigma \nabla u) = 0 \quad \text{in} \quad \Omega$$

Inject currents on $\partial\Omega$: $$g = \sigma \frac{\partial u}{\partial \nu}$$

Measure potentials on $\partial\Omega$: $$f = u$$

- **EIT :**
  given (f, g), recover electrical conductivity $\sigma(x)$

General Principle of DSM

- **Define on the measurement surface** $\Gamma = \partial\Omega$

$$\langle \chi, \phi \rangle_{\gamma,\Gamma} := \langle (-\Delta_{\partial\Omega})^{\gamma} \chi, \phi \rangle \quad \forall\, \chi \in H^{2\gamma}(\partial\Omega)\,, \phi \in L^2(\partial\Omega)$$

- Select a set of probing functions $\{\eta_{x,d}\}$ in $H^{2\gamma}(\partial\Omega)$:

(1) nearly orthogonal wrt $\langle \cdot\,,\,\cdot \rangle_{\gamma}$, i.e., $\forall\, y \in \Omega \;\&\; d_x, d_y \in \mathbb{R}^n,$

$$K_{d_x,d_y}(x,y) := \frac{\langle \eta_{x,d_x}, \eta_{y,d_y} \rangle_{\gamma}}{|\eta_{x,d_x}|_Y}$$

like a Gaussian

(2) the probing family is fundamental:

$$f - \Lambda_{\sigma_0} g \approx \sum_k a_k \eta_{x_k,d_k} \quad \text{in } \partial\Omega$$

Choice of probing functions

- Define

$$-\Delta w_{x,d} = -d \cdot \nabla \delta_x \quad \text{in} \quad \Omega \, ; \quad \frac{\partial w_{x,d}}{\partial \nu} = 0 \quad \text{on} \quad \partial\Omega$$

- Dipole potential :

$$D_{x,d}(\xi) := c_n \frac{(x-\xi)\cdot d}{|x-\xi|^n} \, , \quad \xi \in \mathbb{R}^n$$

- Set $\varphi_{x,d} = D_{x,d} - w_{x,d}$

$$-\Delta \varphi_{x,d} = 0 \ \text{ in } \Omega \, ; \quad \frac{\partial \varphi_{x,d}}{\partial \nu} = \frac{\partial D_{x,d}}{\partial \nu} \ \text{ on } \partial\Omega$$

- Dipole potential :

$$\eta_{x,d}(\xi) := w_{x,d}(\xi) \, , \quad \xi \in \partial\Omega$$

Probing functions for special geometries

- For 3D spheric measurement surface :

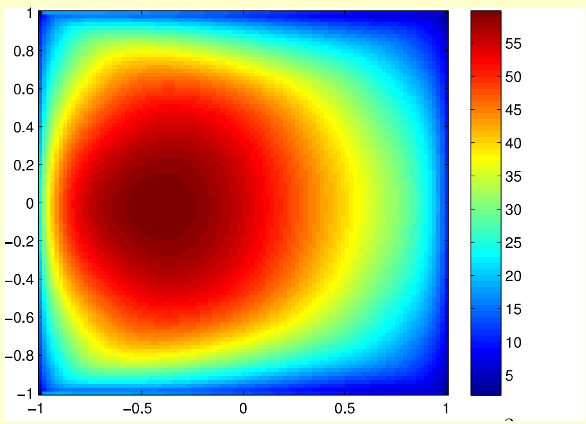
$$\eta_{x,d}(\xi) = \frac{d\cdot\xi - \frac{(x-\xi)\cdot d}{|x-\xi|}}{\sqrt{4\pi}(|x-\xi| - x\cdot\xi + 1)}$$

- For 2D circular measurement curve :

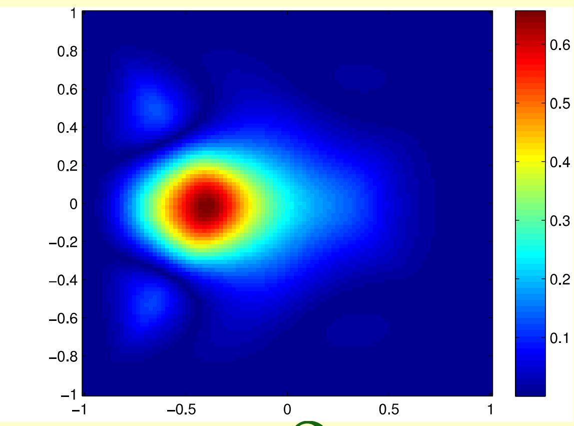
$$\eta_{x,d}(\xi) = \frac{1}{\pi}\frac{(\xi - x)\cdot d}{|x-\xi|^2}$$

Verification of properties

- Can verify orthogonality & fundamental property



$\gamma = 0$



$\gamma = 2$

Index function

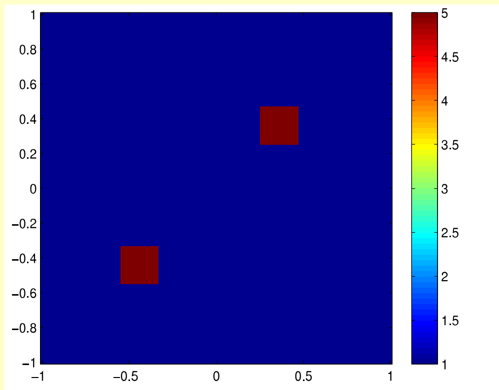
- Define for $x \in \Omega$ & $d_x \in \mathbb{R}^n$,

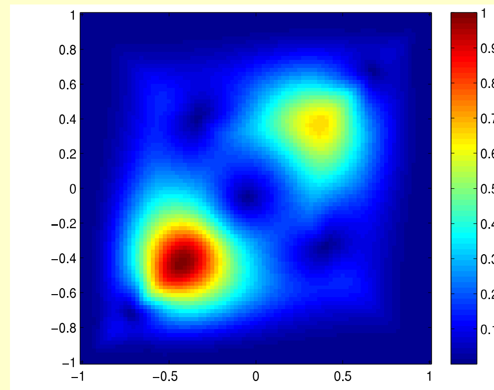
$$I(x, d_x) := \frac{\langle \eta_{x,d_x}, f - \Lambda_{\sigma_0} g \rangle_2}{|\eta_{x,d_x}|_{H^{3/2}(\partial\Omega)} \, || f - \Lambda_{\sigma_0} g ||}$$

Numerical Experiments

- Two separated square objects:

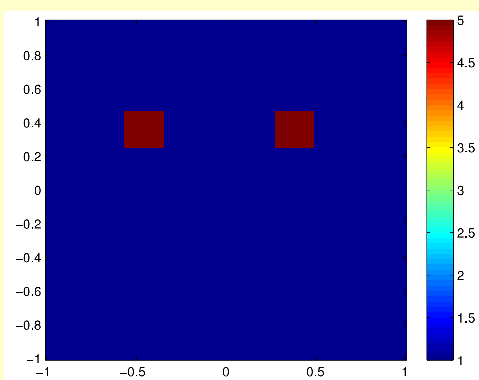
5% noise

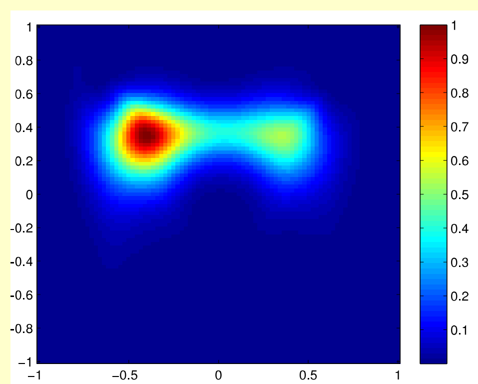




- Two separated square objects

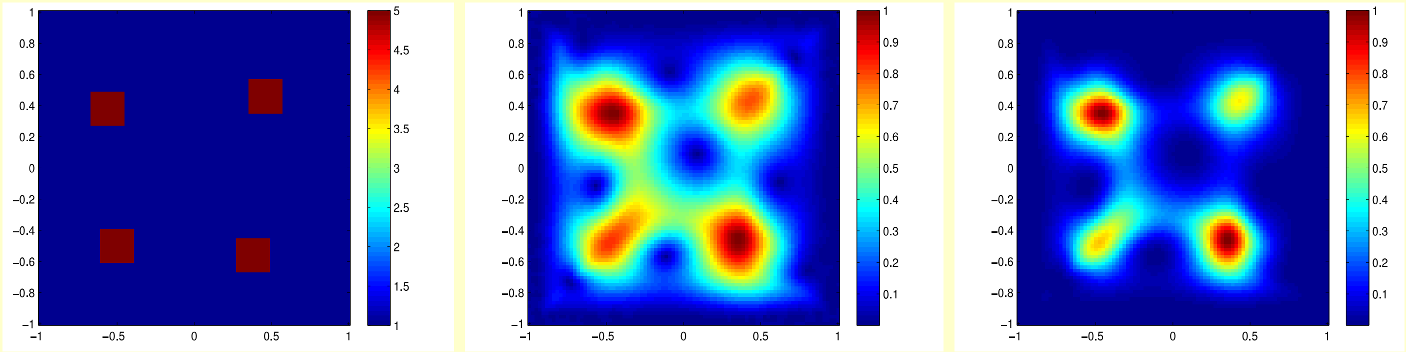
5% noise





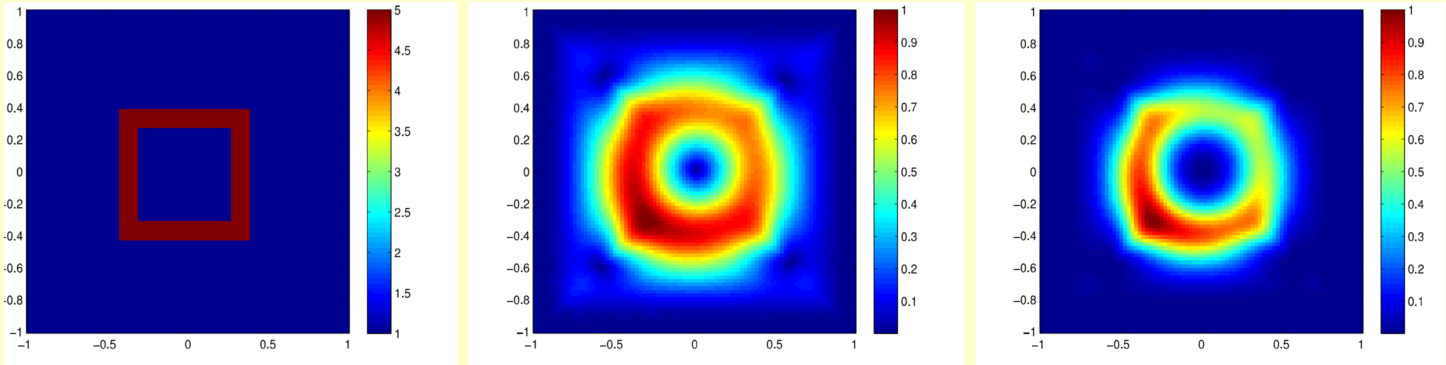
Numerical Experiments

- Four separated square objects



5% noise

- Thin square ring object:



THANK YOU!